Numerical solution of dynamic equilibrium models under Poisson uncertainty[*]

Olaf Posch[(a)] and Timo Trimborn[(b)]

[(a)]Hamburg University and CREATES, [(b)]University of Göttingen

February 2013

Abstract

We propose a simple and powerful numerical algorithm to compute the transition process in continuous-time dynamic equilibrium models with rare events. In this paper we transform the dynamic system of stochastic differential equations into a system of functional differential equations of the retarded type. We apply the Waveform Relaxation algorithm, i.e., we provide a guess of the policy function and solve the resulting system of (deterministic) ordinary differential equations by standard techniques. For parametric restrictions, analytical solutions to the stochastic growth model and a novel solution to Lucas' endogenous growth model under Poisson uncertainty are used to compute the exact numerical error. We show how (potential) catastrophic events such as rare natural disasters substantially affect the economic decisions of households.

JEL classification: C61, E21, O41

Keywords: Continuous-time DSGE, Poisson uncertainty, Waveform Relaxation

1 Introduction

The stochastic growth model in continuous time has received extensive study in the macro literature (following Merton, 1975; Chang and Malliaris, 1987).[1] This benchmark economy gave rise to the development of advanced models for capturing the main features of aggregate fluctuations, often referred to as dynamic stochastic general equilibrium (DSGE) models. These models are the workhorse in dynamic macroeconomic theory. We use them to organize our thoughts, interpret empirical data and for policy recommendations.

The literature on DSGE models, however, has been surprisingly quiet on the effects of large economic shocks such as natural disasters and economic and/or financial crises. Most of the papers focus on small and frequent 'business cycle shocks'. Therefore, departures from Normal uncertainty are largely unexplored. But the simple awareness of large and rare 'Poisson jumps' leads to an adjustment of households' optimal consumption plans. One crucial difference to business cycle shocks is that an econometrician may not observe rare events for a longer period, and thus households might appear to be irrational.

In economic theory, however, we use Poisson events to model, e.g., natural disasters (Barro, 2006), technological improvements (Wälde, 1999, 2005),[2] exploration for exhaustible resources (Quyen, 1991), and financial market bubbles (Miller and Weller, 1990). Similarly, from an empirical perspective, beside anecdotal catastrophic events such as the 2004 Sumatra-Andoman earthquake and tsunami (South Asia), the 2005 Hurricane Katrina (USA) and the recent 2011 Sendai earthquake (Japan), rare disasters are found to have substantial asset pricing and welfare implications (Barro, 2009). Moreover, there is empirical evidence for rare Poisson jumps (positive and negative) in US macro data (Posch, 2009).

For most applications, economists need to rely on numerical methods to compute the solutions to their models. Thus the literature is making a huge effort in developing powerful computational methods (cf. Judd, 1992; Judd and Guu, 1997). Unfortunately, no rigorous treatment of how to solve dynamic equilibrium models under Poisson uncertainty numerically has been provided so far, and the effects of rare events on approximation errors are unknown.[3]

This paper proposes a simple and powerful method for determining the transition process in dynamic equilibrium models under Poisson uncertainty numerically. It turns out that local approximation techniques are not applicable and most global numerical recipes need to account for the specific nature of rare events. We show how to extend existing standard

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[1]The discrete-time one-sector stochastic neoclassical model was pioneered by Brock and Mirman (1972). The mathematical theory of the neoclassical growth model has its origin in Ramsey (1928).

[2]Rare events in the form of Poisson uncertainty also form the basis in quality ladder and matching models (Grossman and Helpman, 1991; Aghion and Howitt, 1992; Lentz and Mortensen, 2008).

[3]Generally most numerical methods are highly accurate locally (cf. Taylor and Uhlig, 1990; Christiano and Fisher, 2000; Schmitt-Grohé and Uribe, 2004; Aruoba, Fernández-Villaverde and Rubio-Ramírez, 2006).

algorithms when we allow for the possibility of rare events.

Our analysis builds on the continuous-time formulation of a stochastic neoclassical growth model based on Merton (1975). We use the continuous-time formulation for two reasons.[4] Firstly, we can easily compute stochastic differentials for transformations based on random variables under Poisson uncertainty. Secondly, for reasonable parametric restrictions we can solve the models by hand and obtain closed-form policy functions which can be used as a point of reference and to compute the exact numerical error.[5] From these benchmark solutions our numerical method is used to explore broader parameterizations. Our idea is to transform the system of stochastic differential equations (SDEs) into a system of functional differential equations of the retarded type (Hale, 1977). We apply the Waveform Relaxation algorithm, i.e., we provide a guess of the policy function and solve the resulting system of (deterministic) ordinary differential equations (ODEs) by standard techniques.

This procedure is applicable to models which imply a dynamic system of controlled SDEs under Poisson uncertainty. The controls are Markov controls in the form of policy functions (cf. Sennewald, 2007). Although our method can also be applied to Normal uncertainty, existing standard procedures can be used for this class of models (cf. Candler, 1999). We therefore do not advocate the use of the Waveform Relaxation algorithm over alternative approaches in all cases and applications. We aim at expanding the set of tools available to researchers by showing how to solve dynamic economies under Poisson uncertainty.

We show that our solution method works. Although the suggested procedure computes the policy functions for the complete state space — even for non-linear solutions — the maximum (absolute) error compared to the exact solutions is very small. A strength of our approach is that existing algorithms are easily extended to allow for Poisson uncertainty. We illustrate our approach for two popular methods computing numerical solutions to dynamic general equilibrium models, i.e., the backward integration (Brunner and Strulik, 2002) and the Relaxation algorithm (Trimborn, Koch and Steger, 2008). From an economic point of view, we find that (potential) large shocks affect optimal consumption and hours strategies.

The structure of the paper is as follows. In Section 2 of this paper, we describe the class of models of interest. In Section 3, we describe the Waveform Relaxation method in detail and discuss alternative approaches. In Section 4, we present two applications. The first is the stochastic growth model with rare disasters. We choose parameterizations that allow for analytical solutions to compute the numerical error. The second is the Lucas model of endogenous growth including a novel analytical solution under Poisson uncertainty. We

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[4]Continuous time models under uncertainty are widely used in economics (for a survey see Wälde, 2011), a continuous-time New Keynesian model is in Fernández-Villaverde, Posch and Rubio-Ramírez (2011).

[5]Analytical solutions for parametric restrictions are frequently used in macro models (Turnovsky, 1993, 2000; Corsetti, 1997; Wälde, 2005, 2011; Turnovsky and Smith, 2006; Posch, 2009).

conclude in Section 5.

2   The macroeconomic theory

This section introduces a broad class of economic models under Poisson uncertainty which can be solved by means of Waveform Relaxation. Our algorithm (presented in Section 3.2) can be used to study transitional dynamics in models under Poisson uncertainty. We show how standard numerical techniques, which compute the optimal time paths of variables, can be extended to allow for Poisson uncertainty, i.e., how they can be used to solve a system of (stochastic) differential equations. A discussion of alternative approaches is provided in Section 3.3. For this purpose, we develop our theoretical framework in Section 2.1, and then present a simple procedure to obtain the (optimal) dynamic system in Section 2.2.

Our motivation stems from the rare disaster literature (Rietz, 1988; Barro, 2006, 2009). Hence, our illustrations are mainly for rare events such as earthquakes or hurricanes which remove a certain fraction of the capital stock. Obviously, our framework is not limited to this particular class of models. For example, infrequent productivity increases are found in the endogenous growth literature (Wälde, 2005). In any case, below we demonstrate that models with rare (but potentially large) economic shocks are conceptionally different from models with smaller shocks, e.g., 'business cycle shocks' resulting from Normal uncertainty. In a nutshell, we show below that the Bellman equation for models under Poisson uncertainty is a *functional* differential equation, while a partial differential equation is arising under Normal uncertainty (cf. Sennewald, 2007; Wälde, 2011).

2.1   The theoretical framework

Consider the following infinite horizon stochastic control problem,

$$\max E \int_0^\infty e^{-\rho t} u(x_t, c_t) dt \quad s.t. \quad dx_t = f(x_t, c_t) dt + g(x_{t-}, c_{t-}) dN_t, \quad x_0 = x, \qquad (1)$$

in which $x_t \in U_x$ denotes a vector of states from the state space $U_x \subseteq \mathbb{R}^{n_x}$, $c_t \in U_c$ denotes a vector of controls from the control region $U_c \subseteq \mathbb{R}^{n_c}$, $u : U_x \times U_c \mapsto \mathbb{R}$, $f : U_x \times U_c \mapsto \mathbb{R}^{n_x}$ are vector functions which ensure concavity and boundedness, $g : U_x \times U_c \mapsto \mathbb{R}^{n_x \times n_x}$ is an $n_x \times n_x$ matrix, and ρ is the rate of time preference. Let N_t denote the n_x vector of (stochastically independent) Poisson processes with arrival rates $\lambda = (\lambda_1, ..., \lambda_{n_x})^\top$.[6] We define $x_{t-} \equiv \lim_{s \to t} x_s$ for $s < t$ as the left-limit at time t such that x_{t-} is the value an instant before a discontinuity (henceforth jump) and $x_t = x_{t-}$ for continuous paths.

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[6]Though there is no conceptional difficulty in extending our analysis to models where the arrival rate is a function of the control and/or the state variable, $\lambda_t = \lambda(x_t, c_t)$, we consider constant arrival rates.

Economically, $u(x_t, c_t)$ specifies the (instantaneous) reward function, $f(x_t, c_t)$ denotes the drift function of the state variables and $g(x_t, c_t)$ is a matrix specifying the jump of state variables if a 'disaster' occurs. If such a rare event of the type i materializes, then $dN_{i,t} = 1$, which affects state variables through the ith column of the matrix $g(x_t, c_t)$.

2.2 Bellman's principle and reduced form descriptions

Closely following Sennewald (2007), choosing an admissible control, $c \in U_c$ and defining $V(x)$ as the (optimal) value function, we obtain the *Bellman equation*

$$\rho V(x) = \max_{c \in U_c} \left\{ u(x, c) + \frac{1}{dt} E_0 dV(x) \right\},$$

which is a necessary condition for optimality. Using Itô's formula (change of variables),

$$
\begin{aligned}
dV(x) &= V_x(x)^\top f(x, c) dt + \sum_{i=1}^{n_x} \left(V(x + g_i) - V(x) \right) dN_i \\
&\equiv V_x(x)^\top f(x, c) dt + v(x, c)^\top dN,
\end{aligned}
$$

in which V_x is the n_x vector of partial derivatives, g_i is the ith column of $g(x, c)$, and $v(x, c)$ stacks the vector of jump terms of the value function corresponding to N. If we take the expectation of the integral form and use the martingale property, assuming that the above integrals exist (Sennewald, 2007, Theorem 2), we arrive at

$$E_0 dV(x) = V_x(x)^\top f(x, c) dt + v(x, c)^\top \lambda dt, \tag{2}$$

and the Bellman equation becomes

$$\rho V(x) = \max_{c \in U_c} \left\{ u(x, c) + V_x(x)^\top f(x, c) + v(x, c)^\top \lambda \right\}.$$

A neat result about the continuous-time formulation (compared to discrete-time models) is that the Bellman equation (2) is, in effect, a deterministic differential equation because the expectation operator disappears (Chang, 2004, p.118). The *first-order conditions* read

$$u_c(x, c) + f_c(x, c)^\top V_x(x) + \sum_{i=1}^{n_x} \left(\partial g_i(x, c) / \partial c \right)^\top V_x(x + g_i) \lambda_i = 0 \tag{3}$$

for any $t \in [0, \infty)$. The first two terms denote the first-order conditions as from deterministic control problems. In case the jump size is a function of the controls, we obtain additional terms represented by the third summand. These terms reflect the effect of the optimal control on the jump size of the states weighted by the probability of arrival. Note that the costate variable is evaluated at different values of the state variables.

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From now on c denotes the optimal control variable. For the *evolution of the costate* we use the maximized Bellman equation,

$$\rho V(x) = u(x,c) + V_x(x)^\top f(x,c) + \sum_{i=1}^{n_x} \left(V(x+g_i) - V(x) \right) \lambda_i. \tag{4}$$

We make use of the envelope theorem to compute the costate,

$$
\begin{aligned}
\rho V_x(x) = {} & u_x(x,c) + f_x(x,c)V_x(x) + V_{xx}(x)^\top f(x,c) \\
& + \sum_{i=1}^{n_x} \left((I + \partial g_i(x,c)/\partial x)^\top V_x(x+g_i) - V_x(x) \right) \lambda_i,
\end{aligned}
$$

in which I denotes the n_x identity matrix. Collecting terms we obtain

$$
\begin{aligned}
\left[\left(\rho + \sum_{i=1}^{n_x} \lambda_i \right) I - f_x(x,c) \right] V_x(x) = {} & u_x(x,c) + V_{xx}(x)^\top f(x,c) \\
& + \sum_{i=1}^{n_x} (I + \partial g_i(x,c)/\partial x)^\top V_x(x+g_i)\lambda_i.
\end{aligned}
\tag{5}
$$

Using Itô's formula, the costate obeys

$$dV_x = V_{xx}(x)^\top f(x,c)dt + \sum_{i=1}^{n_x} \left(V_x(x+g_i) - V_x(x) \right) dN_i,$$

where inserting (5) yields the evolution of the costate variable

$$
\begin{aligned}
dV_x = {} & \left[\left(\rho + \sum_{i=1}^{n_x} \lambda_i \right) I - f_x(x,c) \right] V_x(x)dt - u_x(x,c)dt \\
& - \sum_{i=1}^{n_x} (I + \partial g_i(x,c)/\partial x)^\top V_x(x+g_i)\lambda_i dt + \sum_{i=1}^{n_x} \left(V_x(x+g_i) - V_x(x) \right) dN_i.
\end{aligned}
\tag{6}
$$

The evolution of the costate (6) consists mainly of three parts. Here, λ_i corresponds to the probability that a disaster of the type i occurs over the course of a period Δ, i.e., the probability of one jump during a period Δ is given by $e^{-\lambda_i \Delta}\lambda_i \Delta$. For $\lambda_i = 0$ $(\forall i = 1, \ldots, n_x)$ the costate evolves as in the standard deterministic model given by the first summand. The second part for $\lambda_i > 0$ illustrates the functional dependence of the costate not only on x, but also on the state variables to which the economy jumps in case a rare disaster of the type i occurs, $x + g_i$. In other words, households take into account that disasters may occur. The last part gives the actual jump terms in case of a disaster of the type i, in which $dN_i = 1$ (N_i simply counts the number of arrivals of type i events).

For the general case, we solve the dynamic system (6) and the constraint in (1) together with the static condition (3) for the variables V_x, x and c.

For the case where $g(x, c) = g(x)$ does not depend on c, it is often possible to obtain *Euler equations* for consumption and eliminate the costate from the dynamic system. This yields a system of stochastic differential equations in x and c only. For this, we assume that the inverse function $c = h(V_x, x)$ for the optimality condition (3) exists and is strictly monotonic in both arguments. Then we may write the dynamic equilibrium system as

$$
\begin{aligned}
dx &= f(x,c)dt + g(x)dN, & \text{(7a)} \\
dc &= \frac{\partial h(V_x, x)}{\partial V_x} \left[dV_x - \sum_{i=1}^{n_x} \left(V_x(x + g_i(x)) - V_x(x) \right) dN_i \right] \\
&\quad + \frac{\partial h(V_x, x)}{\partial x} f(x,c)dt + \sum_{i=1}^{n_x} (h(V_x(x + g_i(x)), x + g_i(x)) - h(V_x(x), x)dN_i, & \text{(7b)}
\end{aligned}
$$

in which we insert dV_x from (6) and eliminate V_x using the static condition (3).

Further, the transversality condition is $\lim_{t\to\infty} e^{-\rho t} V(x) \geq 0$ for all admissible paths, where the equality holds for the optimal solution.

3   The numerical solution

This section transforms the reduced form (7) into a system of functional differential equations of the retarded type (RDEs). An extension to the more general case, where the costate variables cannot be eliminated, does not pose any conceptional difficulties.[7] We apply the Waveform Relaxation algorithm, i.e., we provide a guess of the optimal policy function and solve the resulting systems of ODEs.

3.1   Description of the problem

The system of controlled stochastic differential equations (7) can be generalized to

$$
\begin{aligned}
dx_t &= f(x_t, c_t)\, dt + g(x_{t-})dN_t, & \text{(8a)} \\
dc_t &= h(x_t, c_t, [c(x)])\, dt + j(x_{t-}, c_{t-}, [c(x)])dN_t, & \text{(8b)}
\end{aligned}
$$

given initial states x_0 and transversality conditions. By the argument $[c(x)]$ we define the optimal solution or policy function, i.e., the optimal control as a function of a given state, $c = c(x)$. Here, a square bracket, $[\cdot]$, emphasizes the fact that the dependence of h and j on $c(x)$ is non-local, but on the whole policy function. The optimal solution $c(x)$ is of course *a priori* unknown. We assume that system (8) has a unique solution which only depends on the state variables and define the function $c(x) : U_x \subseteq \mathbb{R}^{n_x} \mapsto U_c \subseteq \mathbb{R}^{n_c}$, where U_x denotes

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[7]Note that the proposed procedure handles algebraic equations in a straightforward way.

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the state space and U_c the control region with n_x and n_c denoting the number of states and controls. Similarly, we define the functions f, g, h and j as $f : U_x \times U_c \mapsto \mathbb{R}^{n_x}$, $g : U_x \mapsto \mathbb{R}^{n_x}$, $h : U_x \times U_c \times C^k(U_x, U_c) \mapsto \mathbb{R}^{n_c}$, and $j : U_x \times U_c \times C^k(U_x, U_c) \mapsto \mathbb{R}^{n_c}$, respectively.[8] All functions are of class C^k, i.e., the partial derivatives of up to (and including) order k exist and are continuous with k being sufficiently large.

Consumers' choice of control variables depends on the whole policy function $c(x)$, because they consider the probability that a (Poisson) 'disaster' hits the economy. In this rare event, the state variable x_t jumps by $g(x_{t-})$ and consumption adjusts accordingly. In normal times, however, when no disaster occurs, consumers still consider the possibility that a disaster could occur in the next instant of time for their optimal plans. In equation (6) this is illustrated by the fact that the evolution of the costate depends on the current state *and* on the state of the economy immediately after a disaster occurs. Equation (8b) accounts for this fact by including the complete solution $c(x)$ on the right hand side. Hence, the more general formulation of system (8) includes our system (7), and accounts exactly for this mechanism: hypothetical 'after-disaster' states and controls influence today's decisions.

System (8) has to be augmented by boundary conditions for the beginning and the end of the time horizon. Transversality conditions usually require (scale-adjusted) variables to converge towards some interior steady states for $t \to \infty$, conditional on no jumps, $dN_t \equiv 0$.[9] We denote steady-state values by $\{x^*, c^*\} \subseteq U_x \times U_c$. Generally, it is not sufficient to compute the solution on some subdomain of U_x. The reason can be illustrated by the example of a one-dimensional state space. If the domain is restricted to, e.g., $[x_0, x^*]$ the state could be thrown back to an even smaller value than x_0 or jump to a value above x^*. For that reason, the optimal control on $[x_0, x^*]$ depends on the optimal control for some $x_t < x_0$ and $x_t > x^*$. To take this dependency into account the solution has to be computed on the domain U_x, which for macroeconomic problems typically is $\mathbb{R}_+$. Extending this to an n_x-dimensional state space the solution has to be calculated on the entire domain U_x.

3.2 The Waveform Relaxation algorithm

The crucial task for the numerical solution is to compute the policy function implied by the (conditional) deterministic system, which means for $dN_t \equiv 0$,

$$dx_t = f(x_t, c_t)\, dt, \tag{9a}$$

$$dc_t = h(x_t, c_t, [c(x)])\, dt. \tag{9b}$$

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[8]For cases where one function, say $h = h(x_t, c_t, Z_t, [c(x)])$ is a function of a random variable, Z_t, in general our procedure requires conditioning, $Z_t = z$, such that $h = h(x_t, c_t, z, [c(x)]) \equiv \bar{h}(x_t, c_t, [c(x)])$.

[9]If no ambiguity arises, we use 'steady state' and 'conditional steady state' interchangeably.

In a second step, the stochastic paths are obtained by adding the Poisson process N_t, making use of the policy function $c_t = c(x_t)$. By construction, any solution to (9) solves the Bellman equation (4). The controls and the states follow the paths implied by the system (9) as long as no jump occurs (pathwise continuous). If a jump occurs at date t, the system adjusts according to j and g with $c_{t-} = c(x_{t-})$.

In the mathematical literature, the equations in system (9) are referred to as functional differential equations of the retarded type (cf. Hale, 1977; Kolmanovskii and Myshkis, 1999). A well-known special case of these equations are differential-difference equations (DDEs) in which the dynamic system exhibits a time delay (e.g., Boucekkine, Germain, Licandro and Magnus, 1998, 2001; Asea, Zak 1999). In system (8), the jump term in case of a disaster is known in terms of controls and states, not in terms of time and, hence, the solution methods for DDEs are not suitable. This is why we apply a more general algorithm to solve functional differential equations. Our method is also suitable for solving systems of DDEs.

For calculating the policy function $c_t = c(x_t)$ we exploit the fact that numerous numerical methods are available to solve (9) *without* a dependency on the optimal solution,

$$
\begin{aligned}
dx_t &= f\left(x_t, c_t\right) dt, & \text{(10a)} \\
dc_t &= \tilde{h}\left(x_t, c_t\right) dt. & \text{(10b)}
\end{aligned}
$$

The idea of Waveform Relaxation algorithms is as follows: by providing a guess of the policy function $c_0(x)$, system (9) reduces to (10), because the feedback of the solution path on dc_t is neglected.[10] Now, problem (10) is a standard system of ODEs and can thus be solved by standard algorithms.[11] In general, the obtained solution $c_1(x)$ will be different from the initial guess $c_0(x)$. Hence, a solution of the original (deterministic) problem (9) is not found yet. In the next step the initial guess is updated to $c_1(x)$ and the loop is repeated. If the updated solution $c_i(x)$ is the same as the guess $c_{i-1}(x)$, a solution of the deterministic problem (9) is found (we summarize our Waveform Relaxation algorithm in Table 1).

More formally, we construct a fix-point iteration for the operator $\mathcal{N}$ such that a function z is a fix point of this operator: $\mathcal{N}(z) = z$. The function z represents the desired solution, $z : \mathbb{R} \mapsto \mathbb{R}^{n_x + n_c}$. The operator $\mathcal{N}$ is defined by a modification of problem (9). We start with a trial solution z_0 and iterate by evaluating $\mathcal{N}$, until $\|z_i - z_{i-1}\|$ is sufficiently small.

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[10]Waveform Relaxation algorithms for initial value problems and appropriate error estimation are described in Feldstein, Iserles and Levin (1995), Bjørhus (1994) and Bartoszewski and Kwapisz (2001). Alternative procedures for solving system (9) are collocation methods as described in Bellen and Zennaro (2003).

[11]For problems with one state variable, among others, these are the backward integration procedure (Brunner and Strulik, 2002) and the procedure of time elimination (Mulligan and Sala-i-Martín, 1991). For problems with multiple state variables we can use projection methods (e.g., Judd, 1992), the method of Mercenier and Michel (1994), and the Relaxation method (Trimborn et al., 2008).

Table 1: Summary of the Waveform Relaxation algorithm

Step 1	(Conditioning)	Construct the conditional deterministic system of RDEs (system 9).
Step 2	(Initialization)	Provide an initial guess for the policy function.
Step 3	(Solution)	Solve the resulting system of ODEs (system 10).
Step 4	(Update)	Update the policy function.
Step 5	(Iteration)	Repeat Step 3 and Step 4 until convergence.

For defining the operator $\mathcal{N}$, we take a trial solution $c_0(x)$ as given. We define

$$
\begin{align}
dx_i &= f(x_i, c_i)\, dt, \tag{11a}\\
dc_i &= h(x_i, c_i, c_{i-1}(x))\, dt, \tag{11b}
\end{align}
$$

for each iteration $i = 1, \ldots, n$. Thus, system (11) represent a system of ordinary differential equations which can be solved by the existing standard numerical methods.

For single-state problems ($n_x = 1$) we employ the backward integration method proposed by Brunner and Strulik (2002).[12] For a brief description of this method, recall that equations (11a) and (11b) represent a system of ODEs with an interior, computable stationary point. This point usually exhibits a saddle-point structure, i.e., a stable one-dimensional manifold (policy function) connecting the steady state to the origin, and an unstable one-dimensional manifold. Our task is to compute the stable manifold numerically, for which we exploit the saddle-point structure. By reversing time, the stable (unstable) manifold becomes an unstable (stable) manifold. Thus, by starting near the manifold, solution trajectories are attracted by the (optimal) policy function.

An important difference to standard methods in each iteration step is the evaluation of the function $c_i(x)$. Note that the solution of the previous iteration $c_{i-1}(x)$ is only available at certain points in the phase space. However, functions f and h of system (11) also need an evaluation of $c_{i-1}(x)$ at interior points. We employ a cubic spline interpolation of $c_{i-1}(x)$ to evaluate f and h. In order to control for the improvement in convergence, a suitable norm has to be chosen. We calculate the deviation of the policy function between two iterations on a mesh of points representing the whole state space U_x and employ the Euclidian norm $\|c_i(x_i) - c_{i-1}(x_i)\|$, where $x_i(1) < \ldots < x_i(M)$ and M denotes the number of points on the mesh (M determines the accuracy of the solution).

For multiple-state problems ($n_x > 1$) we employ the Relaxation algorithm as described in Trimborn et al. (2008) to solve the deterministic system (11). This method can be applied to continuous-time deterministic problems with any number of state variables. The principle

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[12]Note that backward iteration can be applied to any number of control variables, i.e., $n_c \geq 1$.

of relaxation is to construct a large set of non-linear equations, the solution of which represents the desired trajectory. This is achieved by a discretization of the involved differential equations on a mesh of points in time. The set of differential equations is augmented by algebraic equations representing equilibrium conditions or (static) no-arbitrage conditions at each mesh point. Finally, equations representing the initial and final boundary conditions are appended. The whole set of equations is solved simultaneously.

For multiple-state problems, we select starting values x_0 uniformly located in a rectangle in the state space U_x and calculate transitional dynamics starting from each of these initial values. The solutions give a good representation of the policy function. Again, the policy function is only available on a mesh in the state space. Similar to the simulations with a one-dimensional state space, we employ a cubic spline interpolation to obtain the policy function at arbitrary interior points. Different from the procedure above, we use only the initial value of each iteration for interpolation. This turns out to be a more robust approach, presumably due to the evenly spaced grid one obtains in this case.

Similar to other procedures, complexity and computation time increases considerably with the number of state variables, which is known as the 'curse of dimensionality'. The computational speed could be improved substantially, however, by using graphics processing units (cf. Aldrich, Fernández-Villaverde, Gallant and Rubio-Ramírez, 2011). The idea is to parallelize the algorithm and employ graphic processors to solve each independent step at the same time. It is straightforward to parallelize the Waveform relaxation algorithm: each of the transition paths starting at a particular initial position in the state space can be used as one independent step and, hence, these paths can be calculated at the same time. Thus, by parallelization, computational costs per iteration can be reduced considerably.

3.3 Comparison to alternative approaches

We briefly compare the Waveform Relaxation algorithm to alternative solution methods which are frequently used in order to solve DGSE models. For a detailed description of these methods see, for example, Judd (1998) and Marimon and Scott (1999).

3.3.1 Local approximation methods

Local approximation methods are widely applied in economics since they are known to solve the stochastic models (under Normal uncertainty) efficiently. The effects of large economic shocks on the approximation error, however, are largely unexplored. This is unfortunate since the local approximation might be inaccurate far off the stationary state around which the solution is approximated. Even without observing large economic shocks, as shown above, households' decisions near (or at) the steady state depend on the value of state variables off

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the steady state, which again would lead to approximation errors since the solution technique would incorrectly incorporate the effect of a potential disaster. The approximation error far off the steady state thus propagates to the steady state region, which implies that local methods might also approximate poorly the policy function around the steady state. This property makes local numerical approximation techniques unattractive: a Poisson shock may drive the economy far away from its steady state value (cf. Barro, 2006), i.e., it may do so in regions where the accuracy is poor. For this reason, local methods are not suitable for approximations of the policy functions in our models.

3.3.2 Global approximation techniques

A number of methods are customized to solve models with Normal uncertainty. Most of these methods exploit the specific structure of the Bellman equation and thus are not suitable for our problem. For example, Büttler (1995) and Candler (1999) apply finite differences to solve the Bellman equation as a partial differential equation (PDE). However, finite differences cannot be used to solve our functional differential equation. As illustrated before, the major difficulty in using such methods is that the value function depends on the current state *and* on the state of the economy immediately after a Poisson shock occurs.

We are aware of at least two global approaches capable of solving our problem at hand: policy function (and value function) iteration and projection methods (spectral methods). Both approaches solve for the policy function numerically using the standard contraction mapping theorem which is independent from its particular functional form. One caveat is that such methods may converge to a wrong solution since the unstable manifold also solves the Bellman equation. Moreover, even for the standard stochastic growth model a sophisticated initial guess for the policy function is needed in order to obtain the correct solution and/or to achieve convergence (cf. Judd, 1992, p.431). Our approach is not subject to those limitations. Since the Waveform Relaxation algorithm solves the system of ODEs for the time path of variables instead of solving the Bellman equation, by construction, it converges to the stable manifold.[13] Hence, we do not encounter problems of convergence towards the unstable manifold even when starting with an uninformative initial guess.

To summarize, the Waveform relaxation algorithm is a global and non-linear solution method and scores with robustness and reliability. A detailed comparison to alternative approaches along the lines of Taylor and Uhlig (1990) is on our research agenda.

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[13]By solving for the time path of variables, the solution satisfies initial boundary conditions for time t_0 and final boundary conditions in the form of steady state values for time ∞. Since our solution converges towards steady state values as $t \to \infty$, it characterizes the stable manifold.

4   Illustrative examples

The following examples are intended to illustrate potential economic applications in macro.
To start with, we first consider the stochastic Ramsey problem with a single control and state
variable, and then use a stochastic version of the Lucas model of endogenous growth mainly
to illustrate the fact that multi-dimensional systems do not pose conceptional difficulties. In
order to keep notation simple, we only consider problems faced by a benevolent planner, and
use capital letters to denote variables in the planning problem which correspond to individual
variables in the household's and firm's problems.

4.1   A neoclassical growth model with disasters

This section solves the stochastic neoclassical growth model under Poisson uncertainty which
is motivated by the Barro-Rietz rare disaster hypothesis (Rietz, 1988; Barro, 2006).

Specification. Suppose that production takes the form of Cobb-Douglas, $Y_t = K_t^\alpha L^{1-\alpha}$,
$0 < \alpha < 1$. Labor is supplied inelastically and capital can be accumulated according to

$$dK_t = (Y_t - C_t - \delta K_t)\,dt - \gamma K_{t-}dN_t, \quad (K_0, N_0) \in \mathbb{R}_+^2, \quad 0 < \gamma < 1, \tag{12}$$

where N_t denotes the number of (natural) disasters up to time t, occasionally destroying γ
percent of the capital stock K_t at an arrival rate $\lambda \geq 0$.[14]

The benevolent planner maximizes welfare by choosing the optimal path of consumption,

$$\max_{\{C_t\}_{t=0}^\infty} E \int_0^\infty e^{-\rho t}\frac{C_t^{1-\theta}}{1-\theta}dt \quad s.t. \quad (12). \tag{13}$$

Solution. From the Bellman principle, a necessary condition for optimality is

$$\rho V(K_0) = \max_{C_0 \in \mathbb{R}_+} \left\{ \frac{C_0^{1-\theta}}{1-\theta} + (K_0^\alpha L^{1-\alpha} - C_0 - \delta K_0)V_K(K_0) + (V(K_0 - \gamma K_0) - V(K_0))\lambda \right\},$$

and the first-order condition corresponding to (3) reads

$$C_t^{-\theta} - V_K(K_t) = 0 \tag{14}$$

for any $t \in [0, \infty)$, making the control variable a function of the state variable, $C_t = C(K_t)$.
Hence, the problem (13) can be summarized as a system of controlled SDEs,

$$\begin{aligned}
dK_t &= (K_t^\alpha L^{1-\alpha} - C_t - \delta K_t)dt - \gamma K_{t-}dN_t, \\
dC_t &= \left(\alpha K_t^{\alpha-1}L^{1-\alpha} - \rho - \delta - \lambda + \lambda(1-\gamma)\tilde{C}(K_t)^{-\theta}\right)C_t/\theta\,dt - \left(1 - \tilde{C}(K_{t-})\right)C_{t-}dN_t,
\end{aligned}$$

in which we define $\tilde{C}(K_t) \equiv C((1-\gamma)K_t)/C(K_t)$, such that $1-\tilde{C}(K_{t-})$ denotes the percentage
drop of optimal consumption after a disaster.

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[14]For a stochastic neoclassical growth model with elastic labor supply and the asset market implications of the Barro-Rietz rare disaster hypothesis, the interested reader is referred to Posch (2011).

4.1.1   Evaluation of the algorithm

We calculate numerical solutions for two benchmark calibrations. In both cases, an analytical representation of the policy function can be computed for plausible parameter restrictions. Therefore, we can compare numerical and analytical solutions and calculate computational errors to evaluate the performance of the Waveform Relaxation algorithm.[15]

Because the neoclassical growth model has one state variable, it is well suited for the backward integration procedure (cf. Brunner and Strulik, 2002). As explained above, by starting near the steady-state value K^* (the value towards which the economy converges if no disasters occur), the solution trajectories are attracted by the optimal policy function.[16]

Our first benchmark solution employs a plausible parameterization which allows for an analytical solution $(\alpha, \theta, \delta, \lambda, \gamma) = (0.5, 2.5, 0.05, 0.2, 0.1)$ similar to Posch (2009) and imposes $\rho = ((1-\gamma)^{1-\alpha\theta} - 1)\lambda - (1-\alpha\theta)\delta$ which gives $\rho = 0.0178$. Using this parametrization, the average time between two disasters is $1/\lambda = 5$ years, with each Poisson event destroying 10 percent of the capital stock. For less frequent and/or smaller rare events, as e.g. for US data (with γ roughly 2.5 percent), our algorithm would improve performance since the true solution is closer to the deterministic guess. As shown in the appendix, consumers choose a constant saving rate $s \equiv 1/\theta$ and the policy function is $C_t = C(K_t) = (1-s)L^{1-\alpha}K_t^\alpha$. Thus the optimal jump term is constant, $\tilde{C}(K_t) = (1-\gamma)^\alpha$. Although technically a knife-edge solution, the policy functions for solutions around this parameter region are very similar. As shown in Figure 1, the deterministic policy function (for $\lambda = 0$ and/or $\gamma = 0$) and the stochastic policy function differ substantially for our calibration. This illustrates that (potential) rare events can have substantial effects on households' behavior.

Figures 2a and 2b show the absolute and relative error of the numerically obtained policy function compared to the analytical solution, respectively. Both plots indicate that the solution exhibits a high accuracy even for a large deviation from the steady state implied by economically large shocks. The absolute and relative errors compared to the true solution are below 10^{-8} within the most relevant interval between 0 and K^*. The maximum (absolute) errors are below 10^{-5} for values of capital of 150 percent of K^*, which is below the accuracy usually required for economic applications. Economically, this value denotes the error as a fraction of consumption at K_t: with a relative error of 10^{-5}, the consumer is making a \$1 mistake for each \$10,000 spent (Aruoba et al., 2006, p.2499).

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[15]The literature typically evaluates the performance using Euler equation residuals (see e.g. Judd, 1992). Santos (2000) shows that approximation errors of the policy function are of the same order of magnitude as the Euler equation residuals. Hence, we are able to compare our results with algorithms solving similar models (as in Aruoba, Fernández-Villaverde and Rubio-Ramírez, 2006; Dorofeenko, Lee and Salyer, 2010).

[16]For the backward integration procedure we deviate 10^{-12} in magnitude from the 'steady state' and we choose 10^{-14} as relative error tolerance for the Runge-Kutta procedure (cf. Brunner and Strulik, 2002).

Figures 2c and 2d show the absolute and relative change of the policy function, respectively, compared to the previous iteration. It is apparent that both functions are of the same shape and order of magnitude as the numerical errors compared to the analytical solution. This shows that the change of the policy function between two iterations is an excellent approximation for measuring the numerical error of the solution. We make use of this striking similarity to define our criterion function to gauge the accuracy of the numerical solution for the general case where no analytical solution is available.

Our second benchmark solution requires the parametric restriction $\alpha = \theta$, which implies a linear policy function, $C_t = C(K_t) = \phi K_t$.[17] As shown in the appendix, the marginal propensity to consume is $\phi = (\rho - ((1-\gamma)^{1-\theta} - 1)\lambda - (\theta-1)\delta)/\theta$. Since the policy function is linear, the optimal jump term is constant, $\tilde{C}(K_t) = 1 - \gamma$. For ease of comparison, we choose the same calibration for parameters as above, but a smaller value for the parameter of relative risk aversion (or higher value for the intertemporal elasticity of substitution), $\theta = 0.5$. As shown in Figure 3a both the deterministic policy function and stochastic policy function are indeed linear in the capital stock. Once again, both policy functions differ substantially. Figure 3b shows the optimal jump in consumption with respect to capital, which is again independent of capital.

Figures 4a and 4b show the absolute and relative error of the numerically obtained policy function compared to the analytical solution, respectively. In fact, the solution is obtained already at the first iteration and exhibits a high accuracy of roughly 10^{-15}, close to the machine's precision. The reason for immediate convergence and high accuracy lies in the linearity of deterministic and stochastic policy functions: jump terms computed from a linear policy function are independent from the policy function's slope and are thus already correctly computed at the first iteration. Figures 4c and 4d show the absolute and relative change of the policy function, respectively, compared to the previous iteration. Again both measures are of similar shape and order of magnitude.

In order to show that the accuracy of the algorithm does not depend on the chosen parameters, we conduct a sensitivity analysis with respect to the frequency of disasters, λ, and the percentage of capital that is destroyed by a disaster, γ. Parameter sets 1 and 4, summarized in Table 2, reiterate our two benchmark calibrations. Parameter sets 2 and 5 assume that a higher proportion of the capital stock is destroyed if a disaster occurs ($\gamma = 0.2$), but such disasters occur with a lower probability ($\lambda = 0.1$). Finally, parameter sets 3 and 6 assume an even higher destruction rate ($\gamma = 0.4$) and a lower probability ($\lambda = 0.05$) of disasters. The parameters are chosen such that an analytical solution is available. For the parameter sets 1 to 3 we adjust the rate of time preference such that

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[17]This solution is well established in macroeconomics (cf. Posch, 2009, and the references therein).

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Table 2: Accuracy of the solution to the neoclassical growth model with disasters

Parameter set[a]	Number of iterations	absolute deviation (last iteration) $\|y_n - y_{n-1}\|$	relative deviation (last iteration) $\|y_n/y_{n-1} - 1\|$	absolute deviation (analytical solution) $\|y_n - y\|$	relative deviation (analytical solution) $\|y_n/y - 1\|$
1	15	$6.7 \cdot 10^{-6}$	$5.2 \cdot 10^{-5}$	$6.7 \cdot 10^{-6}$	$1.2 \cdot 10^{-5}$
2	19	$1.3 \cdot 10^{-6}$	$1.9 \cdot 10^{-4}$	$8.7 \cdot 10^{-7}$	$1.3 \cdot 10^{-6}$
3	28	$2.3 \cdot 10^{-9}$	$6.5 \cdot 10^{-6}$	$1.6 \cdot 10^{-9}$	$9.0 \cdot 10^{-7}$
4	2	$2.1 \cdot 10^{-13}$	$1.1 \cdot 10^{-13}$	$1.7 \cdot 10^{-13}$	$1.0 \cdot 10^{-13}$
5	2	$1.1 \cdot 10^{-13}$	$7.5 \cdot 10^{-14}$	$1.2 \cdot 10^{-13}$	$8.0 \cdot 10^{-14}$
6	2	$1.2 \cdot 10^{-13}$	$1.6 \cdot 10^{-13}$	$1.3 \cdot 10^{-13}$	$1.3 \cdot 10^{-13}$

[a] Parameter sets used in the sensitivity analysis:

1: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 2.5, 0.05, \mathbf{0.2}, \mathbf{0.1}, 0.0178)$,   2: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 2.5, 0.05, \mathbf{0.1}, \mathbf{0.2}, 0.0182)$,
3: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 2.5, 0.05, \mathbf{0.05}, \mathbf{0.4}, 0.0193)$,   4: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 0.5, 0.05, \mathbf{0.2}, \mathbf{0.1}, 0.0178)$,
5: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 0.5, 0.05, \mathbf{0.1}, \mathbf{0.2}, 0.0178)$, 6: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 0.5, 0.05, \mathbf{0.05}, \mathbf{0.4}, 0.0178)$.

$\rho = ((1-\gamma)^{1-\alpha\theta} - 1)\lambda - (1-\alpha\theta)\delta$. For the parameter sets 4 to 6 we use $\alpha = \theta$. Our results for parameter sets 1 to 3 show that the more severe disasters are (the more capital is destroyed), the more iterations are necessary, while the accuracy of the algorithm remains high. Results for parameter sets 4 to 6 show that convergence is indeed immediate and the accuracy almost at the machine's precision for linear policy functions.

Our third illustration in Figure 5a shows both the deterministic and the stochastic policy functions for the intermediate case of logarithmic preferences, $\theta = 1$, for which no analytical solution is known. As shown in Figure 5b the optimal jump term now indeed varies with the capital stock and the function $\tilde{C}(K_t)$ is decreasing in capital. As before, we iterate until convergence, i.e., the change of the policy function between two iterations is sufficiently small (cf. Figures 5a and 5b). Because no analytical benchmark solution is available, we now use that both the absolute and relative change of the policy function between two iterations have the same order of magnitude to conclude that the maximum (absolute) error is roughly 10^{-8} within values for capital between 0 and 150 percent of K^*.

Finally, we should emphasize three main points: First, convergence does not depend on parameter restrictions. The algorithm proves to be stable for a wide range of parameters. We restrict the presentation of results only due to lack of space. Second, computational requirements are rather small. The solution of the model on a standard laptop requires between some seconds and a few minutes. Third, our procedure can be implemented with an average ability in computational skills. While the numerical solution of the deterministic system is standard, the novel part of it is an interpolation routine based on the Waveform Relaxation idea. However, most software packages provide routines for (spline) interpolation. The Matlab codes and details of our implementation are summarized in a technical appendix, both are available on request.

4.1.2 The economic effects of rare disasters

Asking whether rare disasters lead to higher saving is equivalent to examining whether more uncertainty raises or lowers the marginal propensity to consume. It is well established that the intertemporal substitution effect depresses the marginal propensity to save for risk-averse individuals. The optimum way to maintain the original utility level when uncertainty increases is to consume more today (and thus avoid facing the disaster risk). In contrast, the income effect is a precautionary savings effect, as higher uncertainty implies a higher probability of low consumption tomorrow against which consumers will protect themselves the more, by consuming less, the more averse they are to intertemporal fluctuations of consumption (cf. Leland, 1968; Sandmo, 1970). By using a nonlinear production technology, the neoclassical theory of growth under uncertainty offers a third channel through which uncertainty has effects on the asymptotic distribution of capital (cf. Merton, 1975).

As shown in Weil (1990), the effect on optimal consumption (or saving) depends on the magnitude of the intertemporal elasticity of substitution, $1/\theta$.[18] Moreover, optimal consumption depends on the degree of curvature of the production technology, α, since the curvature of the policy function matters for effective risk aversion (cf. Posch, 2011). In case the income effect is relatively small, $\theta < 1$, the presence of rare disasters tends towards higher consumption (cf. Figure 3a). For the case where income and substitution effects balance each other, $\theta = 1$, the only effect on consumption is due to the concave production technology which depresses the marginal propensity to save (cf. Figure 5a), i.e., the mean capital stock decreases. It is only when the intertemporal elasticity of substitution is small, $\theta > 1$, the precautionary savings motive dominates the substitution effect and eventually the effect of the nonlinear production technology, and savings increase (cf. Figure 1a).

4.2 Lucas' model of endogenous growth with disasters

This section uses the Waveform Relaxation algorithm to solve a stochastic version of the Lucas (1988) endogenous growth model with two controls and two state variables. Motivated by the rare disaster hypothesis, rare events – such as natural disasters – occasionally destroy a fraction of the physical capital stock. Our solution method sheds light on the effects on optimal consumption, human capital accumulation, and thus the balanced growth rate.

Specification. Consider a closed economy with competitive markets, with identical agents and a Cobb-Douglas technology, $y_t = k_t^\alpha (u_t h_t)^{1-\alpha}$, where $0 < \alpha < 1$. Suppose at date t, workers (normalized to one) have skill level h_t and own the physical capital stock, k_t. A

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[18]Weil (1990) shows that risk aversion, by determining the amplitude of the associated reduction in the certainty equivalent rate of return to saving, only affects the magnitude of the effects described above.

worker devotes u_t of his non-leisure time to current production, and the remaining $1 - u_t$ to human capital accumulation (improving skills). Hence, the effective aggregate hours devoted to production are $u_t h_t$. Denoting w_t as the hourly wage rate per unit of effective labor, the individual's labor income at skill h_t is $w_t h_t u_t$. Let the rental rate of physical capital be r_t. For simplicity there is no capital depreciation such that k_t evolves according to

$$dk_t = (r_t k_t + w_t u_t h_t - c_t)\,dt - \gamma k_{t-} dN_t, \qquad (16)$$

where N_t denotes the number of (natural) disasters up to time t, occasionally destroying $0 < \gamma < 1$ percent of the capital stock k_t at an arrival rate $\lambda \geq 0$.

To complete the model, the research effort $1 - u_t$ devoted to the accumulation of human capital must be linked to h_t. Suppose the technology relating the change of human capital dh_t to the level already attained and the effort devoted to acquiring more is

$$dh_t = (1 - u_t)\vartheta h_t dt. \qquad (17)$$

According to (17), if no effort is devoted to human capital accumulation, $u_t = 1$, then non accumulates. If all effort is devoted to this purpose, $u_t = 0$, h_t grows at rate $\vartheta > 0$. In between these extremes, there are no diminishing returns to the stock h_t.

The resource allocation problem faced by the representative individual is to choose a time path for c_t and for u_t in $U_c \subseteq \mathbb{R}_+ \times [0,1]$ such as to maximize expected life-time utility,

$$\max_{\{c_t, u_t\}_{t=0}^{\infty}} E_0 \int_0^{\infty} e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \quad s.t. \quad (16) \text{ and } (17), \quad (k_0, h_0, N_0) \in \mathbb{R}_+^3, \qquad (18)$$

where $\theta > 0$ denotes constant relative risk aversion and ρ is the subjective time preference.

Solution. From the Bellman principle, choosing the controls $c_0, u_0 \in U_c$ requires the *Bellman equation* as a necessary condition for optimality,

$$\rho V(k_0, h_0) = \max_{c_0, u_0 \in U_c} \left\{ c_0^{1-\theta}/(1-\theta) + (r_0 k_0 + w_0 u_0 h_0 - c_0) V_k + (1 - u_0)\vartheta h_0 V_h \right.$$
$$\left. + (V((1-\gamma)k_0, h_0) - V(k_0, h_0))\lambda \right\}. \qquad (19)$$

For any $t \in (0, \infty)$, the two *first-order conditions* corresponding to (3) are

$$c_t^{-\theta} - V_k = 0, \qquad (20)$$
$$w_t h_t V_k - \vartheta h_t V_h = 0, \qquad (21)$$

making the controls a function of the state variables, $c_t = c(k_t, h_t)$ and $u_t = u(k_t, h_t)$.

After some tedious algebra we obtain the Euler equations for consumption and hours. Together with initial and transversality conditions and constraints in (16) and (17), these

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describe the equilibrium dynamics. We may summarize the reduced form dynamics by defining $\tilde{c}(k_t, h_t) \equiv c((1-\gamma)k_t, h_t)/c(k_t, h_t)$ and $\tilde{u}(k_t, h_t) \equiv u((1-\gamma)k_t, h_t)/u(k_t, h_t)$ as

$$dk_t = (r_t k_t + w_t u_t h_t - c_t)\, dt - \gamma k_{t-} dN_t, \tag{22a}$$

$$dh_t = (1 - u_t)\vartheta h_t dt, \tag{22b}$$

$$dc_t = \frac{r_t - \rho - \lambda + \tilde{c}(k_t, h_t)^{-\theta}(1-\gamma)\lambda}{\theta} c_t dt + (\tilde{c}(k_{t-}, h_{t-}) - 1)c_{t-} dN_t, \tag{22c}$$

$$du_t = \left(\tfrac{1-\alpha}{\alpha}\vartheta + \left(\tilde{u}(k_t, h_t)^{-\alpha} - (1-\gamma)^{1-\alpha}\right)(1-\gamma)^{\alpha}\lambda\tilde{c}(k_t, h_t)^{-\theta}/\alpha - c_t/k_t + u_t\vartheta\right)u_t dt$$
$$+ (\tilde{u}(k_t, h_t) - 1)u_{t-} dN_t, \tag{22d}$$

and the transversality condition reads (cf. Benhabib and Perli, 1994, p.117)

$$\lim_{t \to \infty} E_0 \left[V_k e^{-\rho t}[k_t - k_t^*] + V_h e^{-\rho t}[h_t - h_t^*]\right] \geq 0$$

for all admissible k_t and h_t, where k_t^* and h_t^* denote the optimal state values.

Balanced growth. From the reduced form system, we can derive the balanced growth rate of physical capital, human capital and consumption of the conditional deterministic system (conditioned on no disasters) as follows. First, we can neglect the stochastic integrals because for the case with no disasters $dN_t \equiv 0$. Second, similar to the deterministic model, the condition optimal research effort is constant, such that $du_t = 0$ must hold.

Now, for $du_t = 0$ research effort along the balanced growth path is implicitly given by $-\vartheta u^* = \frac{1-\alpha}{\alpha}\vartheta + \left(\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha}\right)(1-\gamma)^{\alpha}\lambda\tilde{c}^{-\theta}/\alpha - c/k$, where $\tilde{u} \equiv \tilde{u}(k_t, h_t)$, $\tilde{c} \equiv \tilde{c}(k_t, h_t)$ and $c/k \equiv c_t/k_t$ are constants. This property of the jump terms implies that asymptotically, $\tilde{c}(k_t, h_t) = \tilde{c}(k_t/h_t)$. Similarly, along this balanced growth path the other equations imply

$$g^k = r^*/\alpha - c/k, \quad g^h = (1 - u^*)\vartheta, \quad g^c = \left(r^* - \rho - \lambda + \tilde{c}^{-\theta}(1-\gamma)\lambda\right)/\theta.$$

Since c_t/k_t is constant, c_t and k_t must grow at the same rate, $g^k = g^c$, which in turn implies $c/k = \left(r^* - \rho - \lambda + \tilde{c}^{-\theta}(1-\gamma)\lambda\right)/\theta + r^*/\alpha$. Along this path we need r^* to be constant, which requires that k_t and h_t grow at the same rate, $g^k = g^h$. Hence, $r^*/\alpha - c/k = (1 - u^*)\vartheta$. This pins down the interest rate $r^* = \vartheta + \left(\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha}\right)\tilde{c}^{-\theta}(1-\gamma)^{\alpha}\lambda$. Hence, the balanced growth rate of the *conditional* deterministic system is

$$g \equiv \left(\vartheta - \rho - \lambda + (1-\gamma)^{\alpha}\tilde{u}^{-\alpha}\tilde{c}^{-\theta}\lambda\right)/\theta, \tag{23}$$

which finally implies the consumption-to-capital ratio

$$c/k = \left(\vartheta + \left(\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha}\right)\tilde{c}^{-\theta}(1-\gamma)^{\alpha}\lambda - \rho - \lambda + \tilde{c}^{-\theta}(1-\gamma)\lambda\right)/\theta$$
$$+ \left(\vartheta + \left(\tilde{u}^{-\alpha} - (1-\gamma)^{1-\alpha}\right)\tilde{c}^{-\theta}(1-\gamma)^{\alpha}\lambda\right)/\alpha.$$

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The growing variables of the reduced-form system c_t, h_t, and k_t in (22) need to be scaled such that they approach some stationary steady-state values (scale-adjustment).

Scale-adjusted dynamics. In what follows, we simply subtract the endogenous balanced growth rate (23) from the reduced-form system in instantaneous growth rates to obtain scale-adjusted variables. The scale-adjusted system (conditioned on no disasters) reads

$$d \ln k_t = (r_t + w_t u_t h_t / k_t - c_t / k_t - g)\, dt, \tag{24a}$$

$$d \ln h_t = (\vartheta - u_t \vartheta - g) dt, \tag{24b}$$

$$d \ln c_t = \left(\left(r_t - \rho - \lambda + \tilde{c}(k_t, h_t)^{-\theta}(1 - \gamma) \lambda \right) / \theta - g \right) dt, \tag{24c}$$

$$du_t = \left(\tfrac{1-\alpha}{\alpha} \vartheta + \left(\tilde{u}(k_t, h_t)^{-\alpha} - (1 - \gamma)^{1-\alpha} \right) (1 - \gamma)^{\alpha} \lambda \tilde{c}(k_t, h_t)^{-\theta} / \alpha - c_t / k_t \right) u_t dt$$
$$+ \vartheta u_t^2 dt, \tag{24d}$$

where g follows iteratively from (23).

Note that in general it is not possible to compute the steady state levels in terms of variables k^*, h^*, c^*, and u^* from system (24). We presume that the stochastic model inherits this characteristic from its deterministic counterpart, which exhibits a ray of steady states, i.e., a center manifold of stationary equilibria (cf. Lucas, 1988; Caballé and Santos, 1993). Each point on this ray differs with respect to the level of physical and human capital and, hence, consumption the economy can generate. The particular stationary equilibrium, to which the economy finally converges is determined by the initial values of physical and human capital. Since in general the functions $\tilde{c}$ and $\tilde{u}$ are not known for the stochastic counterpart of the model, we are not able to prove this property for the general case. However, for the parametric restriction $\alpha = \theta$ we obtain a closed-form solution and indeed provide a proof of this property below. Moreover, our numerical results confirm that the stochastic model indeed exhibits a ray of steady states. A 'steady-state' value in the stochastic setup again refers to the value the economy converges if no disasters occur.

We are now prepared to solve this (scale-adjusted) system using the Relaxation algorithm together with the Waveform relaxation idea.

4.2.1 Evaluation of the algorithm

We calculate numerical solutions for the Lucas model employing a benchmark calibration for which an analytical solution is available. Again, we compare the numerical and analytical solutions to evaluate the algorithm's accuracy. Moreover, we calculate numerical solutions for a second calibration for which no analytical solution is available.

Because this model has two state variables, we choose the Relaxation algorithm to solve system (10) (cf. Trimborn et al., 2008). As already mentioned this algorithm is capable of

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solving deterministic systems with multiple state variables. Moreover, the algorithm can also solve models that exhibit a center manifold of stationary equilibria. Since the method calculates the solution path as a whole, the particular conditional steady state to which the economy converges is determined numerically.

Our benchmark solution uses the calibration $(\alpha, \vartheta, \lambda, \gamma, \rho) = (0.75, 0.075, 0.2, 0.1, 0.03)$ and the parametric restriction $\theta = \alpha$. As shown in the appendix, in this case consumers optimally choose constant hours, $u_t = u = (\rho - (1 - \theta)\vartheta)/(\alpha\vartheta)$, and optimal consumption does not depend on human capital and is linear in physical capital, $c_t = c(k_t, h_t) = \varphi k_t$. $\varphi = (\rho - ((1 - \gamma)^{1-\theta} - 1)\lambda)/\theta$ denotes the marginal propensity to consume with respect to physical capital. Since the policy function is linear in physical capital, the optimal jump terms are constant, $\tilde{c}(k_t, h_t) = 1 - \gamma$ and trivially $\tilde{u}(k_t, h_t) = 1$. Observe that this solution is very similar to the neoclassical growth model, though the growth rate is endogenous. From (23) we find that for $\alpha = \theta$, the balanced growth rate (in normal times, after the transition) is not affected by the presence of rare events, $g = (\vartheta - \rho)/\theta$. Below we compare our numerical solution obtained by the Waveform Relaxation algorithm with the analytical solution.

Figures 7a and 7b, respectively, show the optimal level of consumption and the optimal jump in consumption with respect to physical capital and human capital. Note that the optimal jump in consumption is independent of both physical capital and human capital. Similar to the neoclassical growth model, we find that the deterministic policy function for consumption (for $\lambda = 0$ and/or $\gamma = 0$) and the stochastic counterpart differ substantially. Moreover, the center manifold of stationary equilibria of (scale-adjusted) values for human capital and physical capital is different from the deterministic model.

Figures 8a and 8b show the absolute and relative error of consumption for the computed mesh grid of physical and human capital. Given the nature of the problem, the (absolute) errors are extremely small, not exceeding 10^{-8} in magnitude. As explained above, this level of accuracy is higher than what is usually required for most economic applications. Figures 8c and 8d show the absolute and relative change in the policy function for consumption, respectively, compared to the previous iteration. It is apparent that both functions are of the same shape and order of magnitude as the numerical errors compared to the analytical solution, which helps us to gauge the numerical error of the solution in the general case.

Similarly to the case of consumption, Figures 9a and 9b show the optimal level of hours worked and the optimal jump with respect to physical capital and human capital. Hours are independent of capital goods along the transition and, hence, do not adjust in case of a Poisson jump. Figures 10a and 10b show the absolute and relative error of hours worked, whereas the absolute and relative change in the policy function for hours compared to the previous iteration are shown in Figures 10c and 10d, respectively. Again, the maximum

20

542 (absolute) errors are very small and do not exceed 10^{-6}.

543 As an illustration for a case where closed-form solutions are not available, we compare our
544 benchmark solution to the case of logarithmic preferences, $\theta = 1$. Figures 11 and 13 show the
545 optimal policy functions for consumption and hours and the optimal jump in consumption
546 and hours, respectively. We find that the optimal levels and their jump terms now depend
547 on the level of physical capital and human capital. While the level of optimal consumption
548 is increasing in both capital goods, hours are increasing in human capital but decreasing
549 in physical capital. Hence, countries with an abundant supply of human capital but scarce
550 supply of physical capital tend to supply the most hours to production.

551 Again we would like to emphasize that we are able to calculate policy functions not only
552 for the parametric restrictions presented above, but for a wider range of parameter values.
553 However, the algorithm is not as stable as for the one-dimensional case and is less precise
554 mainly due to interpolation problems. Eventually, for extreme combination of parameter
555 values, problems of convergence might occur, or at least the procedure needs refinement with
556 respect to the chosen mesh and/or interpolation method. Since our main objective is to show
557 that multiple state variables do not pose conceptional problems for our solution method, we
558 leave this work for future research. The Matlab codes and details of our implementation are
559 summarized in a technical appendix, both available on request.

560 4.2.2 The economic effects of rare disasters

561 The Lucas model of endogenous growth has several channels through which uncertainty
562 enters in the economic decisions, and thus optimal plans will be affected when consumers
563 face more uncertainty. First, uncertainty will affect the consumption/saving decision as in the
564 neoclassical growth model. Second, uncertainty will enter the optimal allocation problem
565 of hours devoted to production and human capital accumulation. Finally, their optimal
566 behavior takes account of the effect on the (conditional) balanced growth path.

567 As shown in Figures 7a and 11a, the level of (scale-adjusted) consumption increases
568 for both calibrations, thus the dominating channel is the intertemporal substitution effect,
569 i.e., to consume more today (and thus avoid facing the disaster risk). In other words, the
570 intertemporal elasticity of substitution is sufficiently elastic to compensate the precautionary
571 savings effect. This is in line with the result from the neoclassical growth model.

572 In this model consumption is no longer the only way to accommodate the presence of risk.
573 From Figure 13a, for the case of logarithmic preferences with $\theta = 1$, we find that optimal
574 hours decrease due to the presence of rare disasters (a level shift). Intuitively, consumers
575 prefer to invest more in human capital accumulation which — in contrast to the physical
576 capital good — is not subject to disaster risk. Though it seems an intuitive response from

an asset pricing perspective, we find that this result cannot be generalized. As from Figure 9a, optimal hours are independent of the disaster risk. Supplying less hours for production also has an income effect, which in the case of $\alpha = \theta$ exactly offsets the previous effect. This example illustrates that it is important to study the effects of uncertainty within a dynamic stochastic general equilibrium (DSGE) model, in order to avoid missing potentially important feedback mechanisms when focusing on partial equilibrium effects only.

As from (23), the balanced growth rate (in normal times, after the transition) depends on the optimal jump terms for both consumption and hours. In our numerical solution for $\theta = 1$, the balanced growth rate of the deterministic system of $(\vartheta - \rho)/\theta = 4.5\%$ increases by roughly 0.2 percentage points to $g = 4.7\%$ due to the presence of rare disasters. An intuitive explanation of this effect is indeed the shift of optimal hours supplied to human capital accumulation, and thus implying a higher growth rate in times without disasters.

5   Conclusion

In this paper we propose a simple and powerful method for determining the transitional dynamics in continuous-time DSGE models under Poisson uncertainty. Our contribution is to show how existing algorithms can be extended with an additional layer when we allow for the possibility of rare events in the form of Poisson uncertainty.

We illustrate the algorithm by computing the stochastic neoclassical growth model and a stochastic version of the Lucas model motivated by the Barro-Rietz rare disaster hypothesis. We use analytical solutions for plausible parametric restrictions as a benchmark in order to address the numerical accuracy. We find that even for non-linear policy functions, the numerical error is extremely small.

From an economic perspective, we show that the simple awareness of the possibility of infrequent large economic shocks affects optimal decisions and thus economic growth. The effect is economically important and thus needs to be explored in future research.

An interesting extension of this paper would be to allow for Normal-Poisson uncertainty. In fact, the idea of Waveform Relaxation is not limited to the class of models under Poisson uncertainty. For models under Normal uncertainty, the Bellman equation reduces to a partial differential equation, which is a specific type of a *functional* differential equation. Studying the numerical properties of these models is part of our research agenda.

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A Analytical benchmark solutions

A.1 An analytical solution to the neoclassical growth model

The idea is to provide a guess of the value function and derive conditions under which both the first-order condition and the maximized Bellman equation hold (cf. Posch, 2009).

Suppose that

$$V(K_t) = \frac{\mathbb{C}_1 K_t^{1-\alpha\theta}}{1 - \alpha\theta}. \tag{25}$$

From (14), optimal consumption per effective worker is a constant fraction of income,

$$C_t^{-\theta} = \mathbb{C}_1 K_t^{-\alpha\theta} \quad \Leftrightarrow \quad C_t = C(K_t) = \mathbb{C}_1^{-1/\theta} K_t^{\alpha}.$$

Now use the maximized Bellman equation together with CRRA utility $u(C_t) = C_t^{1-\theta}/(1-\theta)$ and insert the solution candidate,

$$\rho V(K_t) = \frac{C(K_t)^{1-\theta}}{1 - \theta} + (K_t^{\alpha} L^{1-\alpha} - C(K_t) - \delta K_t) V_K + (V((1-\gamma)K_t) - V(K_t))\lambda,$$

which is equivalent to

$$
\begin{aligned}
(\rho + \lambda)\frac{\mathbb{C}_1 K_t^{1-\alpha\theta}}{1 - \alpha\theta} &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} K_t^{\alpha - \alpha\theta}}{1 - \theta} + \left(K_t^{\alpha} L^{1-\alpha} - \mathbb{C}_1^{-1/\theta} K_t^{\alpha} - \delta K_t \right) \mathbb{C}_1 K_t^{-\alpha\theta} \\
&\quad + \frac{\mathbb{C}_1 K_t^{1-\alpha\theta}}{1 - \alpha\theta}(1-\gamma)^{1-\alpha\theta}\lambda \\
\Leftrightarrow \quad 0 &= \frac{\theta}{1 - \theta}\mathbb{C}_1^{-\frac{1}{\theta}} + L^{1-\alpha} - (\rho + (1-\alpha\theta)\delta + \lambda - (1-\gamma)^{1-\alpha\theta}\lambda)\frac{K_t^{1-\alpha}}{1 - \alpha\theta}.
\end{aligned}
$$

This equation has a solution for $\mathbb{C}_1^{-1/\theta} = (\theta - 1)/\theta L^{1-\alpha}$ and

$$\rho = (1-\gamma)^{1-\alpha\theta}\lambda - \lambda - (1-\alpha\theta)\delta. \tag{26}$$

For reasonable parametric calibrations equation (26) is satisfied. Though being a special case, a Keynesian consumption function could be an admissible policy function for the neoclassical model (cf. also Chang, 1988). Its plausibility is an empirical question.

A.2 An analyical solution to the Lucas model

We start with an educated guess on the value function and derive conditions under which it actually is the unique solution of the optimal stochastic control problem. Suppose that

$$V(k_t, h_t) = \frac{\mathbb{C}_1 k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1 - \theta}. \tag{27}$$

From (20), we obtain that optimal consumption is a linear function in the capital stock

$$c_t^{-\theta} = \mathbb{C}_1 k_t^{-\theta} \quad \Rightarrow \quad c(k_t, h_t) = \mathbb{C}_1^{-\frac{1}{\theta}} k_t. \tag{28}$$

Similarly, from (21) we obtain the optimal share of hours allocated to production, u_t,

$$w_t h_t \mathbb{C}_1 k_t^{-\theta} = \vartheta h_t \mathbb{C}_2 h_t^{-\theta} \quad \Leftrightarrow \quad u(k_t, h_t) = \left(\frac{\vartheta}{(1-\alpha)} \frac{\mathbb{C}_2}{\mathbb{C}_1} h_t^{\alpha-\theta} k_t^{\theta-\alpha} \right)^{-\frac{1}{\alpha}},$$

in which we use $w_t = (1-\alpha) k_t^{\alpha} (u_t h_t)^{-\alpha}$. Observe that for the parametric restriction $\alpha = \theta$, optimal hours allocated to production becomes a constant,

$$\alpha = \theta \quad \Rightarrow \quad u(k_t, h_t) = \left(\frac{\vartheta}{(1-\alpha)} \frac{\mathbb{C}_2}{\mathbb{C}_1} \right)^{-\frac{1}{\alpha}}.$$

Using the maximized Bellman equation, we may write with $r_t = \alpha k_t^{\alpha-1} (u_t h_t)^{1-\alpha}$

$$\rho V(k_t, h_t) = \frac{c(k_t, h_t)^{1-\theta}}{1-\theta} + (k_t^{\alpha} (u(k_t, h_t) h_t)^{1-\alpha} - c(k_t, h_t)) V_k + (1 - u(k_t, h_t)) \vartheta h_t V_h$$
$$+ (V((1-\gamma) k_t, h_t) - V(k_t, h_t)) \lambda.$$

Inserting the guess for the value function gives

$$(\rho + \lambda) \frac{\mathbb{C}_1 k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} = \frac{c(k_t, h_t)^{1-\theta}}{1-\theta} + (k_t^{\alpha} (u(k_t, h_t) h_t)^{1-\alpha} - c(k_t, h_t)) \mathbb{C}_1 k_t^{-\theta}$$
$$+ (1 - u(k_t, h_t)) \vartheta h_t \mathbb{C}_2 h_t^{-\theta} + \frac{\mathbb{C}_1 (1-\gamma)^{1-\theta} k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} \lambda.$$

Now insert the policy function for consumption $c(k_t, h_t)$,

$$(\rho + \lambda) \frac{\mathbb{C}_1 k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} = \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} k_t^{1-\theta}}{1-\theta} + \left(k_t^{\alpha} (u(k_t, h_t) h_t)^{1-\alpha} - \mathbb{C}_1^{-\frac{1}{\theta}} k_t \right) \mathbb{C}_1 k_t^{-\theta}$$
$$+ (1 - u(k_t, h_t)) \vartheta \mathbb{C}_2 h_t^{1-\theta} + \frac{\mathbb{C}_1 (1-\gamma)^{1-\theta} k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} \lambda.$$

Now, we employ the restriction $\theta = \alpha$ such that optimal hours are constant, $u(k_t, h_t) = u$,

$$(\rho + \lambda) \frac{\mathbb{C}_1 k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} = \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} k_t^{1-\theta}}{1-\theta} + \left(u^{1-\alpha} h_t^{1-\alpha} - \mathbb{C}_1^{-\frac{1}{\theta}} k_t^{1-\theta} \right) \mathbb{C}_1$$
$$+ (1 - u) \vartheta \mathbb{C}_2 h_t^{1-\theta} + \frac{\mathbb{C}_1 (1-\gamma)^{1-\theta} k_t^{1-\theta} + \mathbb{C}_2 h_t^{1-\theta}}{1-\theta} \lambda.$$

Collecting terms, we obtain

$$\left(\rho + \lambda - \theta \mathbb{C}_1^{-\frac{1}{\theta}} - (1-\gamma)^{1-\theta} \lambda \right) \mathbb{C}_1 k_t^{1-\theta} =$$
$$\left((1-\theta) u^{1-\alpha} \mathbb{C}_1 + (1-\theta)(1-u) \vartheta \mathbb{C}_2 - (\rho + \lambda) \mathbb{C}_2 + \mathbb{C}_2 \lambda \right) h_t^{1-\theta}.$$

28

Hence, the first constant is pinned down by $\mathbb{C}_1 = (\theta/(\rho + \lambda - (1-\gamma)^{1-\theta}\lambda))^\theta$. Inserting u finally pins down the second constant,

$$
\begin{aligned}
\rho\mathbb{C}_2 &= (1-\theta)u^{1-\alpha}\mathbb{C}_1 + (1-\theta)(1-u)\vartheta\mathbb{C}_2 \\
\Leftrightarrow \frac{\rho - (1-\theta)\vartheta}{(1-\theta)\vartheta} &= \frac{\alpha}{1-\alpha}\left(\frac{\vartheta}{(1-\alpha)}\right)^{-\frac{1}{\alpha}}\mathbb{C}_1^{\frac{1}{\alpha}}\mathbb{C}_2^{-\frac{1}{\alpha}} \\
\Rightarrow \mathbb{C}_2 &= \left(\frac{\alpha\vartheta}{\rho - (1-\theta)\vartheta}\right)^\alpha \frac{1-\alpha}{\vartheta}\left(\frac{\theta}{\rho + \lambda - (1-\gamma)^{1-\theta}\lambda}\right)^\theta.
\end{aligned}
$$

Observe that we solved *not only* for some balanced growth path, but for the whole transition path for a parameter restriction. To summarize, for $\alpha = \theta$ we obtain

$$
c(k_t, h_t) = c(k_t) = \frac{\rho + \lambda - (1-\gamma)^{1-\theta}\lambda}{\theta}k_t, \tag{29}
$$

$$
u(k_t, h_t) = u = \frac{\rho - (1-\theta)\vartheta}{\alpha\vartheta}. \tag{30}
$$

Hence, individuals prefer relatively more consumption (or less investment) but work the same hours compared to the deterministic model for $\alpha = \theta$ (a similar condition for deterministic Hamiltonian dynamic systems is in Ruiz-Tamarit, 2008). Note that the analytical solution to the stochastic extension of the Lucas model is novel.

B   Figures
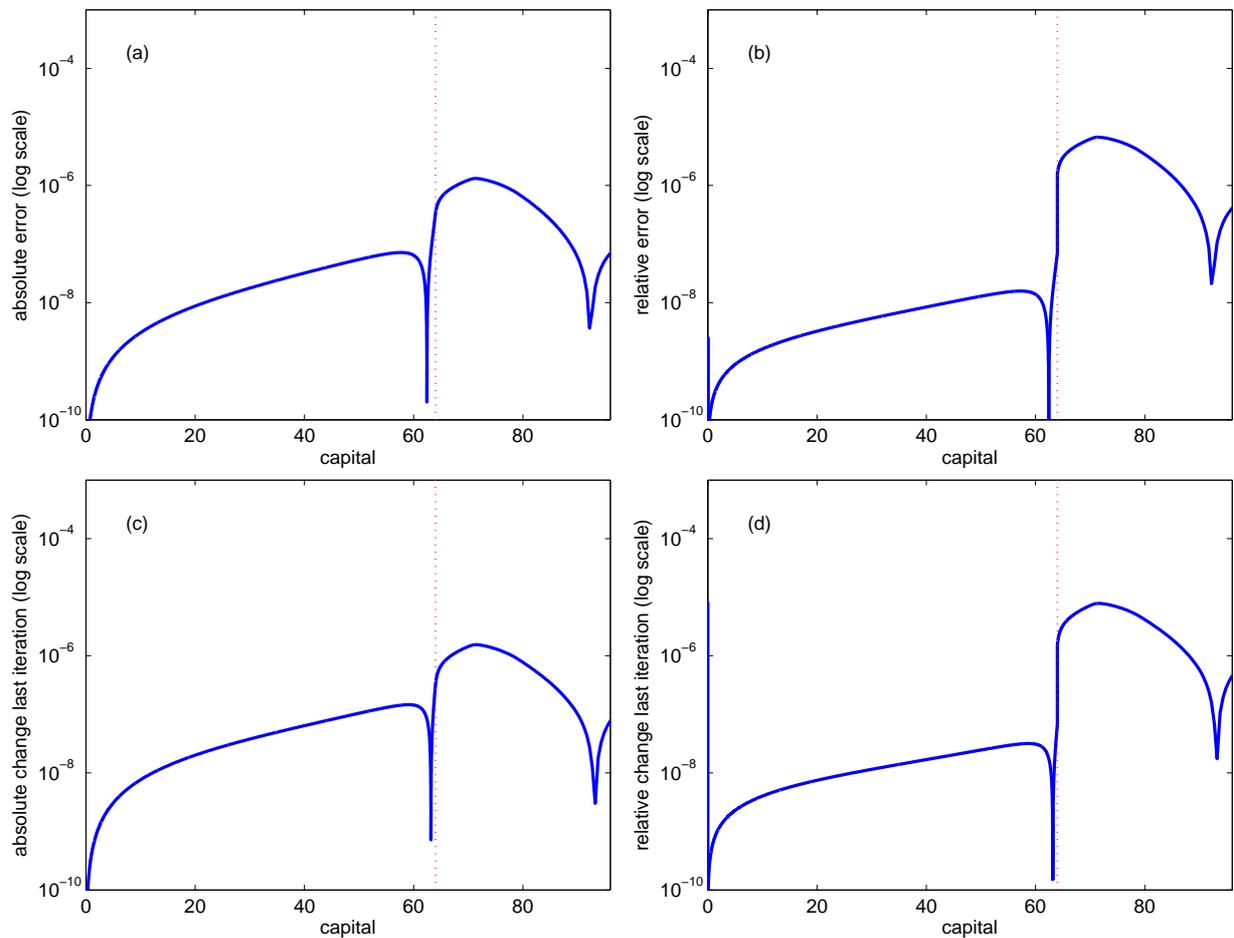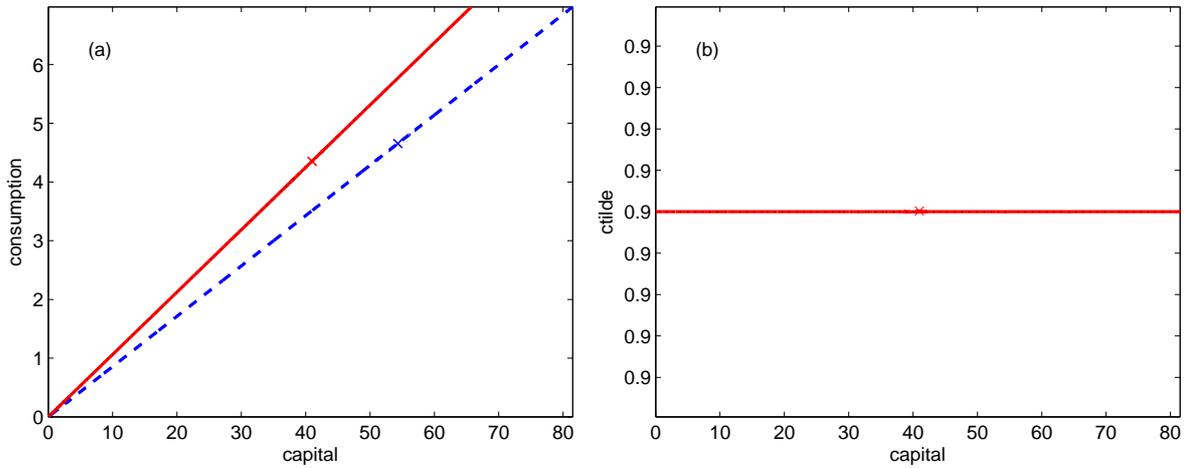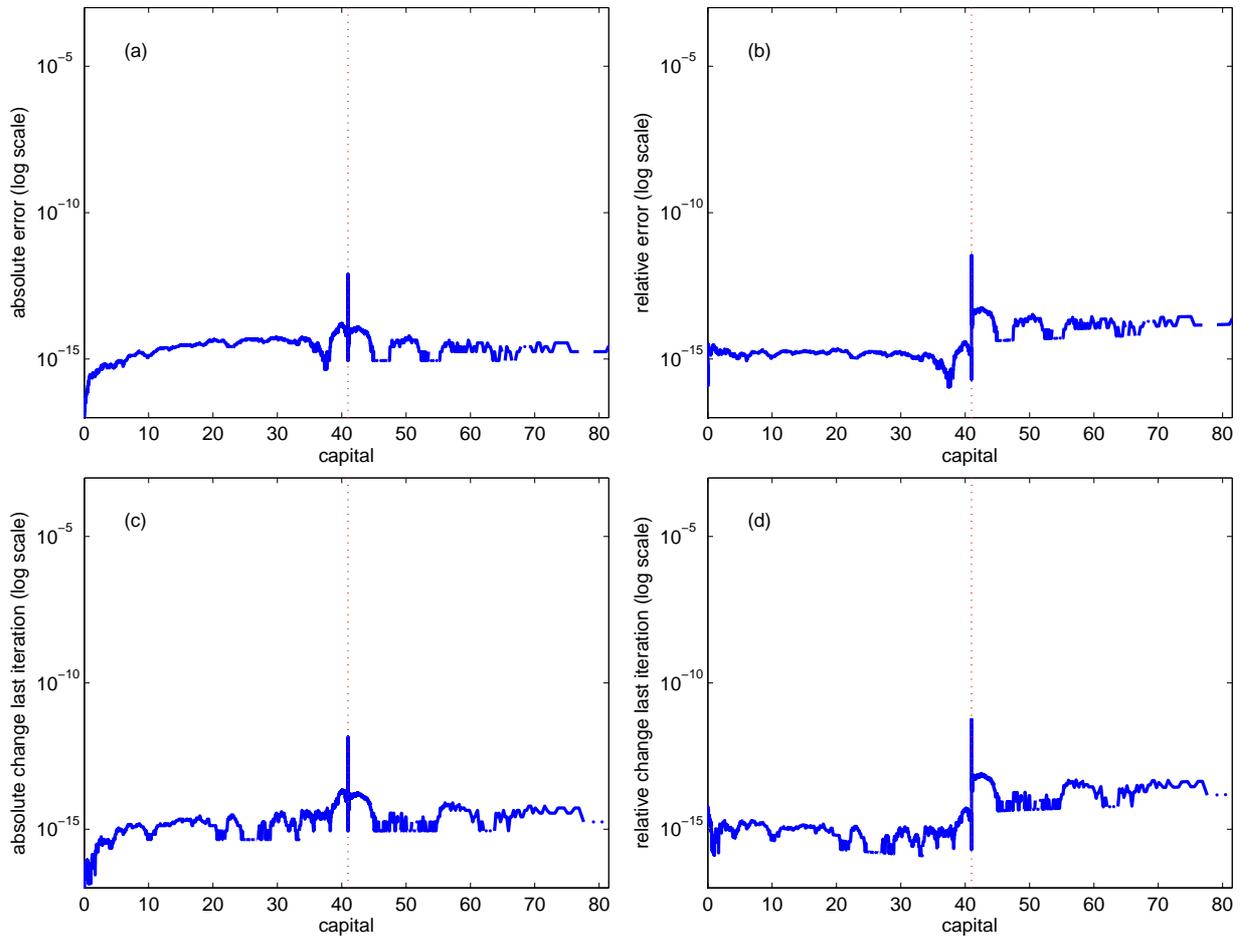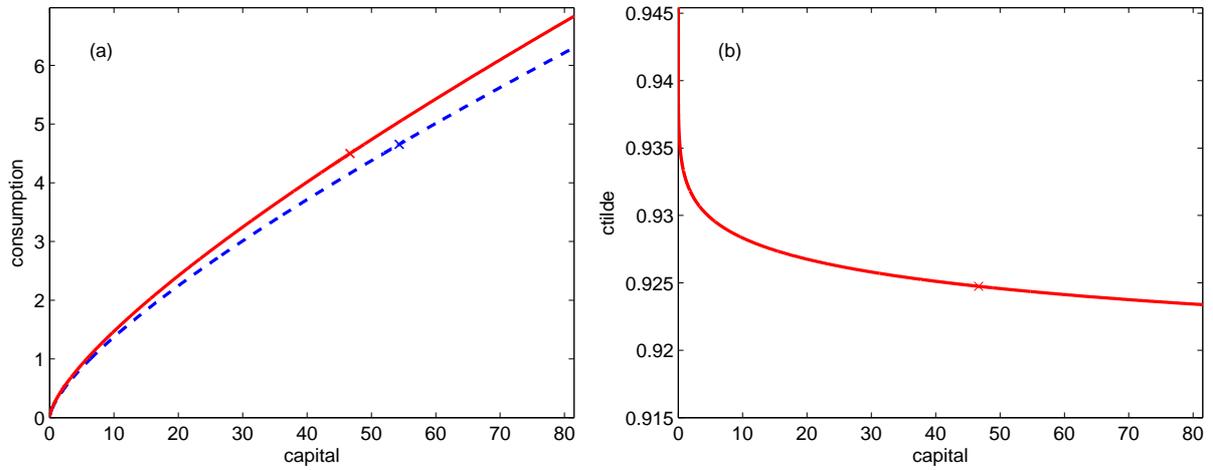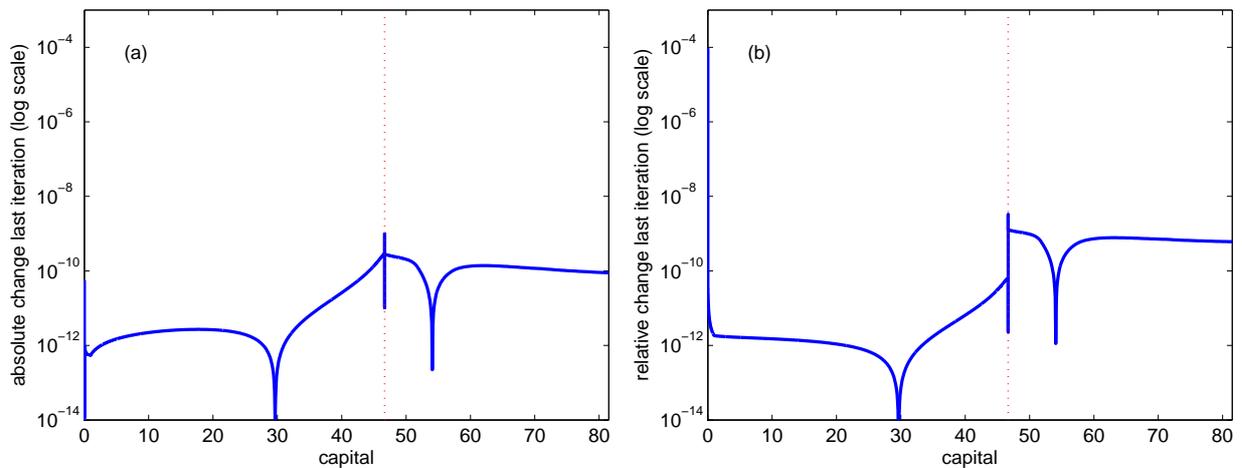
B.1   A neoclassical growth model with disasters

Figure 1: Policy functions and optimal jump in the neoclassical growth model (i)
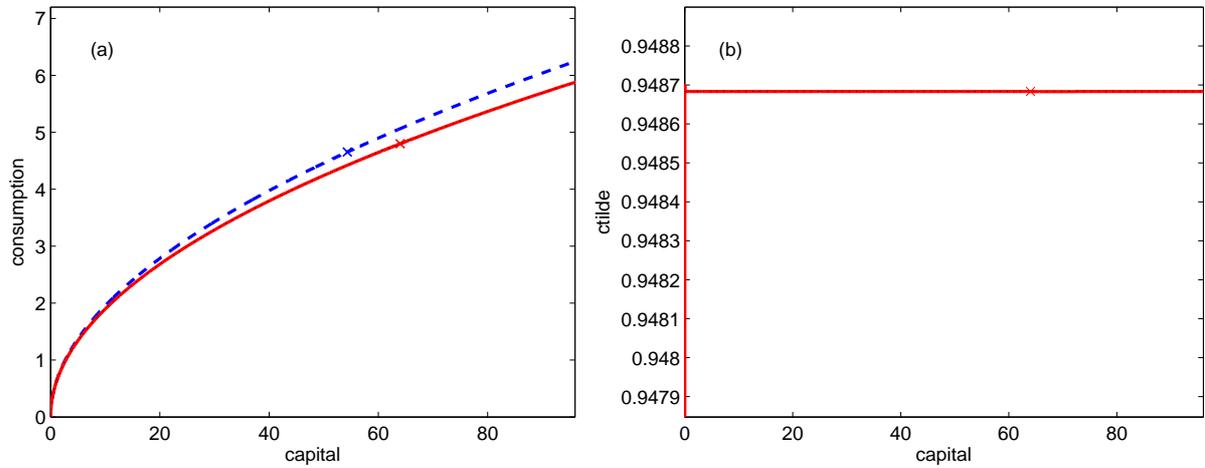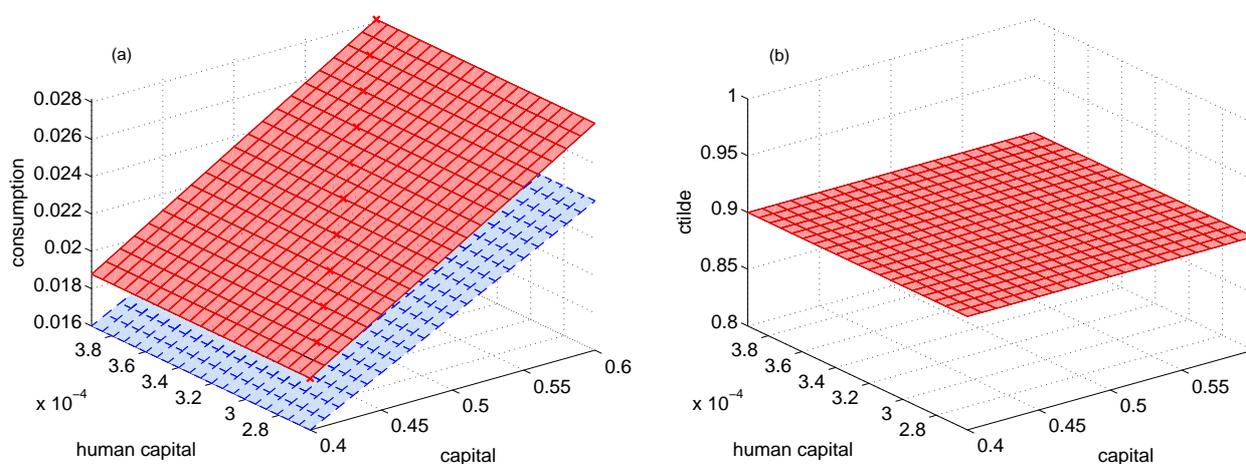

Notes: These figures show (a) the optimal policy functions: deterministic (dashed) vs. stochastic (solid) in the neoclassical growth model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of capital for the parameter set 1 from Table 2: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 2.5, 0.05, 0.2, 0.1, 0.0178)$, which implies a constant saving rate.

Figure 2: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration

Figure 3: Policy functions and optimal jump in the neoclassical growth model (ii)

Notes: These figures show (a) the optimal policy functions: deterministic (dashed) vs. stochastic (solid) in the neoclassical growth model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of capital for the parameter set 4 from Table 2: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 0.5, 0.05, 0.2, 0.1, 0.0178)$, which implies a linear policy function.

Figure 4: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration
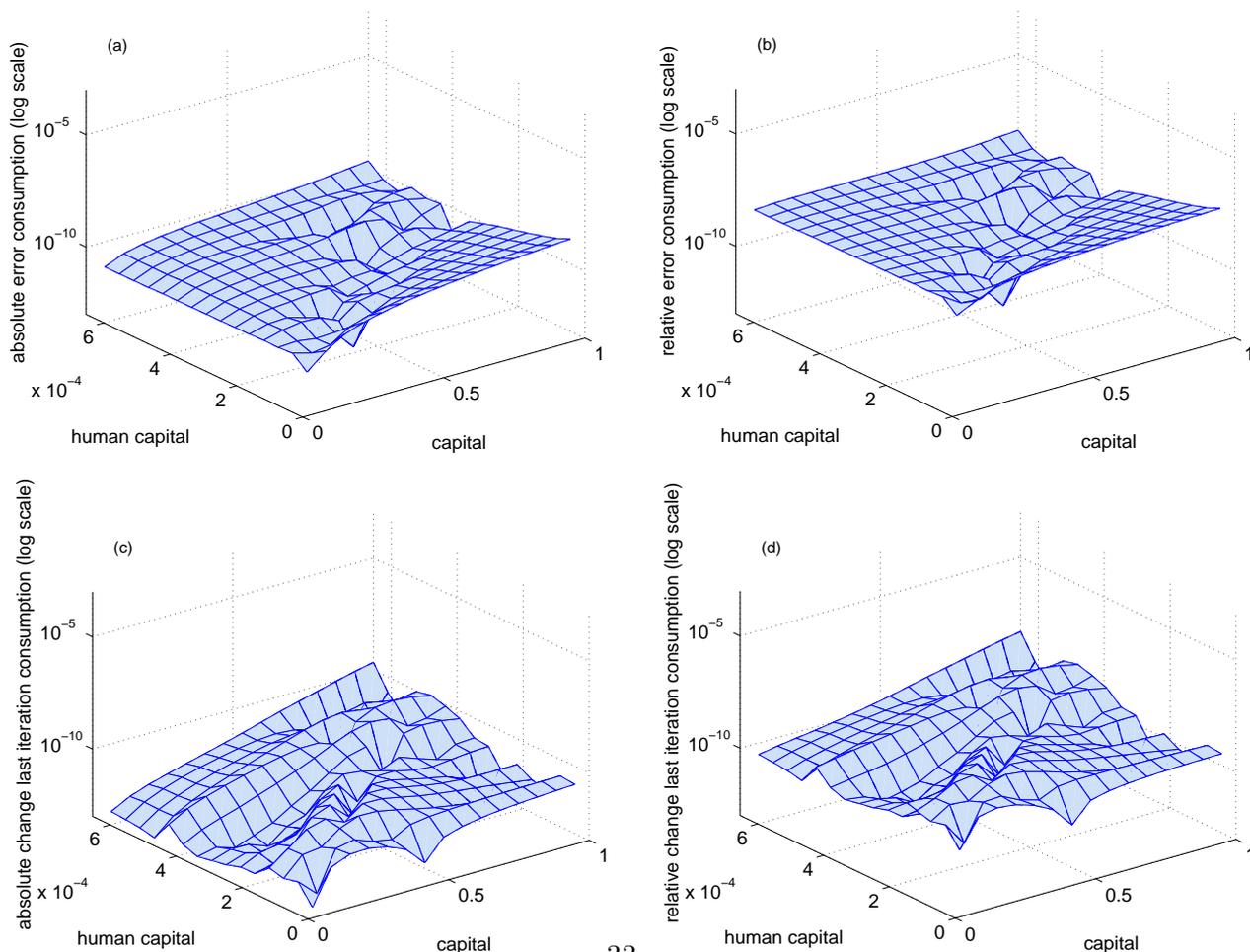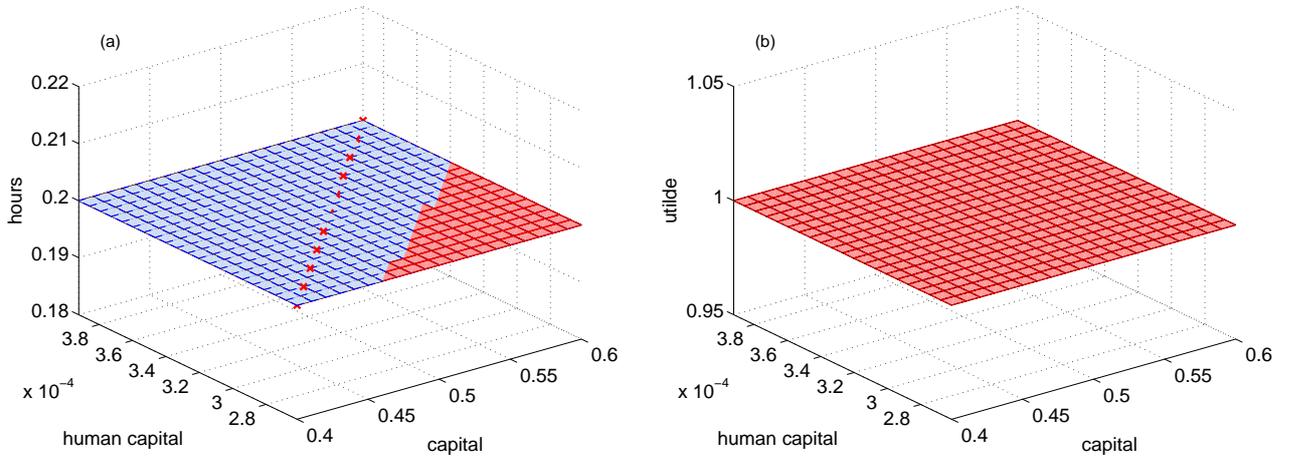
Figure 5: Policy functions and optimal jump in the neoclassical growth model (iii)

Notes: These figures show (a) the optimal policy functions: deterministic (dashed) vs. stochastic (solid) in the neoclassical growth model (no analytical benchmark solution available), and (b) the optimal jump as a function of capital for the parameter set: $(\alpha, \theta, \delta, \lambda, \gamma, \rho) = (0.5, 1, 0.05, 0.2, 0.1, 0.0178)$.

Figure 6: Absolute and relative error compared to the policy function of the last iteration (no analytical errors available)

B.2 Lucas' model of endogenous growth with disasters

Figure 7: Policy functions and optimal jump for consumption in the Lucas model (1)

Notes: These figures show (a) the optimal policy functions: deterministic (dashed) vs. stochastic (solid) in the Lucas model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 0.75, 0.075, 0.2, 0.1, 0.03)$, which implies a linear policy plane.

Figure 8: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration

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Figure 9: Policy functions and optimal jump for hours in the Lucas model (1)

Notes: These figures show (a) the optimal policy functions: deterministic (dashed) vs. stochastic (solid) in the Lucas model compared to the analytical benchmark solution (dotted), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 0.75, 0.075, 0.2, 0.1, 0.03)$, which implies a linear policy plane.

Figure 10: Absolute and relative error compared to the analytical benchmark solution and to the policy function of the last iteration
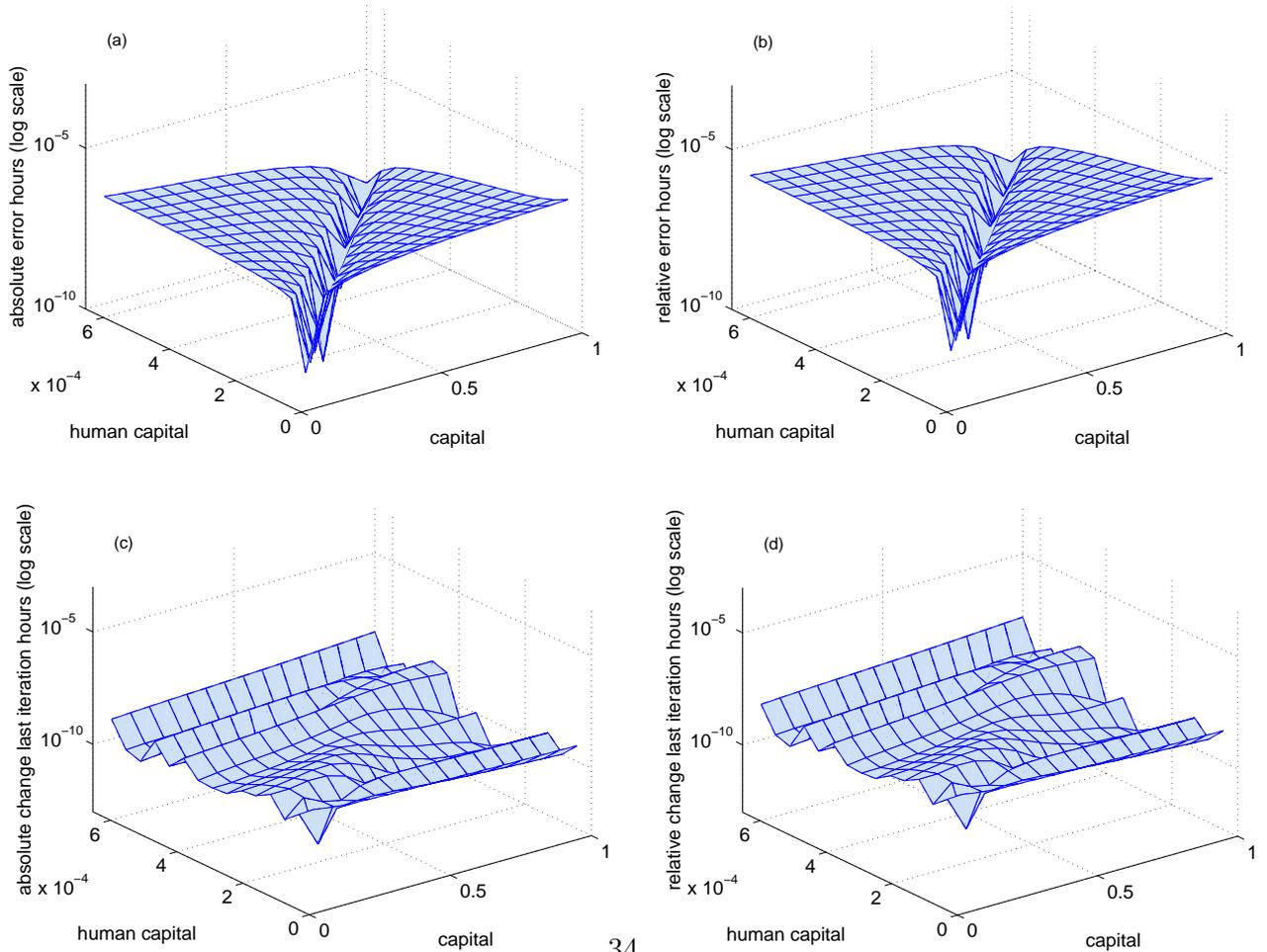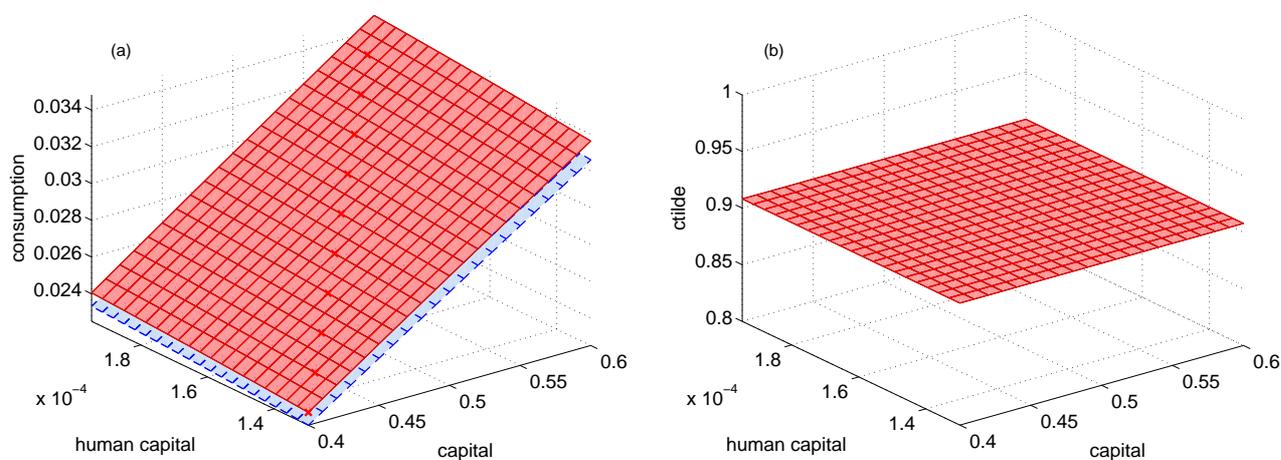
Figure 11: Policy functions and optimal jump for consumption in the Lucas model (2)

Notes: These figures show (a) the optimal policy functions: deterministic (dashed) vs. stochastic (solid) in the Lucas model (no analytical benchmark solution available), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 1, 0.075, 0.2, 0.1, 0.03)$.

Figure 12: Absolute and relative error compared to the policy function of the last iteration (no analytical errors available)
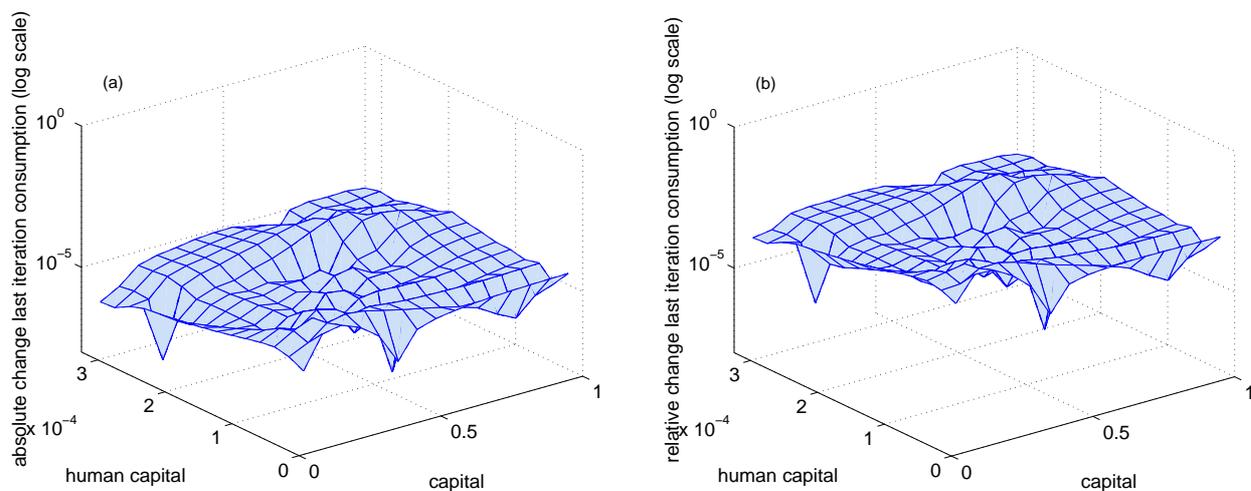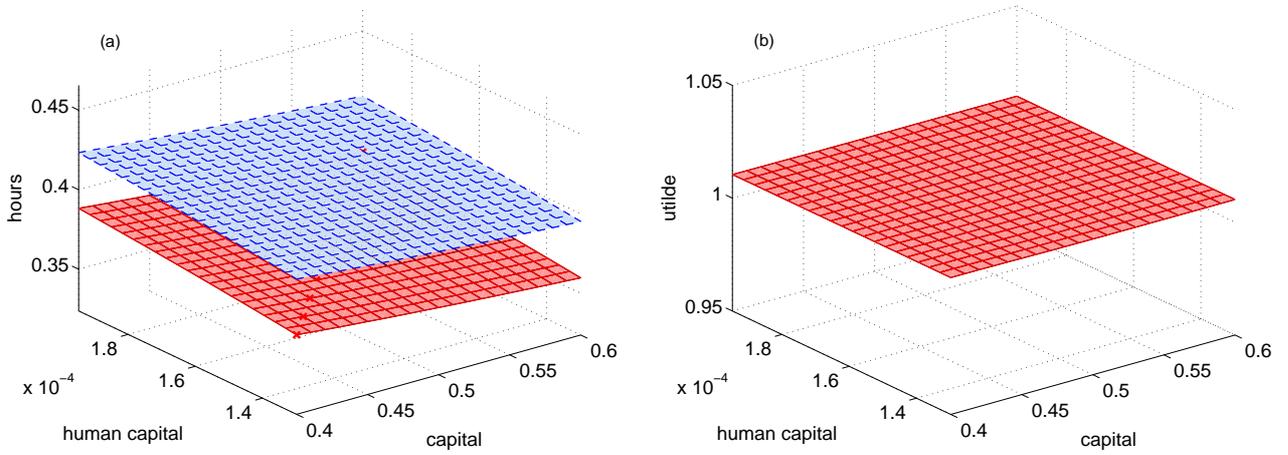
Figure 13: Policy functions and optimal jump for hours in the Lucas model (2)

Notes: These figures show (a) the optimal policy functions: deterministic (dashed) vs. stochastic (solid) in the Lucas model (no analytical benchmark solution available), and (b) the optimal jump as a function of physical capital and human capital for the calibration $(\alpha, \theta, \vartheta, \lambda, \gamma, \rho) = (0.75, 1, 0.075, 0.2, 0.1, 0.03)$.

Figure 14: Absolute and relative error compared to the policy function of the last iteration (no analytical errors available)