Risk Matters: Breaking Certainty Equivalence in Linear Approximations

Online Appendix (not for publication)

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C Definitions and Rules of Matrix Differentiation

The following results and their proofs were introduced in Vetter (1973) and later complemented in Brewer (1978). They have been recently used in Chen and Zadrozny (2003) and Lan and Meyer-Gohde (2013, 2014) for the construction of multidimensional derivative-based approximations of functional problems without the need of multidimensional arrays or tensor notation. Similar results have been applied by Gomme and Klein (2011) and Binning (2013a,b).

Definition 1 (Jabobian matrix)**.** Let $\mathbf{x}$ denote an $n \times 1$ vector. The first-order derivative of the vector-valued function $\mathbf{g}(\mathbf{x}) : \mathbf{x} \to \mathbb{R}^m$ with respect to the vector $\mathbf{x}$ is given by the $m \times n$ Jacobian matrix

$$\mathbf{g}_{\mathbf{x}} := \mathcal{D}_{\mathbf{x}^\top}\{\mathbf{g}(\mathbf{x})\} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}, \tag{C.1}$$

where $^\top$ indicates transposition.

Definition 2 (Matrix derivative)**.** Let $\mathbf{x}$ denote an $n \times 1$ vector. The first-order derivative of the matrix-valued function $\mathbf{G}(\mathbf{x}) : \mathbf{x} \to \mathbb{R}^{m \times n}$ with respect to the vector $\mathbf{x}$ is given by the $m \times n^2$ matrix

$$\mathbf{G}_{\mathbf{x}} := \mathcal{D}_{\mathbf{x}^\top}\{\mathbf{G}(\mathbf{x})\} = \begin{bmatrix} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \end{bmatrix} \otimes \mathbf{G} = \begin{bmatrix} \mathbf{G}_{x_1} & \cdots & \mathbf{G}_{x_n} \end{bmatrix}, \tag{C.2}$$

where $\otimes$ denotes the Kronecker product and $\mathbf{G}_{x_i}$ is the derivative of the matrix-valued function $\mathbf{G}(\mathbf{x})$ with respect to the scalar x_i $(i = 1, \ldots, n)$ given by the $m \times n$ matrix

$$\mathbf{G}_{x_i} := \mathcal{D}_{x_i}\{\mathbf{G}(\mathbf{x})\} = \begin{bmatrix} \frac{\partial G_{1,1}}{\partial x_i} & \cdots & \frac{\partial G_{1,n}}{\partial x_i} \\ \vdots & \vdots & \vdots \\ \frac{\partial G_{m,1}}{\partial x_i} & \cdots & \frac{\partial G_{m,n}}{\partial x_i} \end{bmatrix}.$$

The k-th order derivative of the matrix-valued function $\mathbf{G}(\mathbf{x}) : \mathbf{x} \to \mathbb{R}^{m \times n}$ with respect to the vector $\mathbf{x}$ is given by the $m \times n^m$ matrix

$$\mathbf{G}_{\mathbf{x}^k} := \mathcal{D}_{(\mathbf{x}^\top)^k}\{\mathbf{G}(\mathbf{x})\} = \left(\begin{bmatrix} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \end{bmatrix}^{\otimes[k]} \right) \otimes \mathbf{G}, \tag{C.3}$$

where $[\ldots]^{\otimes[k]}$ denotes the k-fold Kronecker product, i.e., $\underbrace{[\ldots] \otimes \cdots \otimes [\ldots]}_{k \text{ times}}$.

Definition 3 (Multidimensional calculus)**.** Let $\mathbf{M}(\mathbf{x})$, $\mathbf{N}(\mathbf{x})$ and $\mathbf{P}(\mathbf{x})$ be, respectively, $p \times q$, $q \times u$ and $u \times v$ matrix-valued functions of the $n \times 1$ vector $\mathbf{x}$. Moreover, let $\mathbf{Q}(\mathbf{w}(\mathbf{x}))$

be a $p \times q$ matrix-valued function of the $u \times 1$ vector-valued function $\mathbf{w}(\mathbf{x})$. Finally, let $\mathbf{R}(\mathbf{Z}(\mathbf{Y}))$ be a $p \times q$ matrix-valued function of the $u \times v$ matrix-valued function $\mathbf{Z}(\mathbf{Y})$, where $\mathbf{Y}$ is an $s \times t$ matrix. Then, using definitions (C.1)-(C.3) the following results for multidimensional calculus hold:

1. Matrix product rule:

$$\mathcal{D}_{\mathbf{x}^\top}\{\mathbf{M}(\mathbf{x})\,\mathbf{N}(\mathbf{x})\} = \mathcal{D}_{\mathbf{x}^\top}\{\mathbf{M}(\mathbf{x})\}\,(\mathbf{I}_n \otimes \mathbf{N}(\mathbf{x})) + \mathbf{M}(\mathbf{x})\,\mathcal{D}_{\mathbf{x}^\top}\{\mathbf{N}(\mathbf{x})\}, \tag{C.4}$$

   where $\mathbf{I}_n$ is an $n \times n$ identity matrix.

2. Matrix chain rule I:

$$\mathcal{D}_{\mathbf{x}^\top}\{\mathbf{Q}(\mathbf{w}(\mathbf{x}))\} = \mathcal{D}_{\mathbf{w}^\top}\{\mathbf{Q}(\mathbf{w}(\mathbf{x}))\}\,(\mathcal{D}_{\mathbf{x}^\top}\{\mathbf{w}(\mathbf{x})\} \otimes \mathbf{I}_q), \tag{C.5}$$

   where $\mathbf{I}_q$ is a $q \times q$ identity matrix.

3. Matrix chain rule II:

$$\mathcal{D}_{\mathbf{Y}}\{\mathbf{R}(\mathbf{Z}(\mathbf{Y}))\} = \mathcal{D}_{\mathbf{Y}}\{(\operatorname{vec}\mathbf{Z})^\top \otimes \mathbf{I}_p\}\,(\mathbf{I}_t \otimes \mathcal{D}_{\operatorname{vec}\mathbf{Z}}\{\mathbf{R}\}), \tag{C.6}$$

   where $\mathbf{I}_q$ is a $q \times q$ identity matrix, and vec is the vectorization operator that transforms the $u \times v$ matrix $\mathbf{Z}$ into an $uv \times 1$ vector.

4. Matrix Kronecker product rule:

$$\mathcal{D}_{\mathbf{x}^\top}\{\mathbf{M}(\mathbf{x}) \otimes \mathbf{P}(\mathbf{x})\} = \mathcal{D}_{\mathbf{x}^\top}\{\mathbf{M}(\mathbf{x})\} \otimes \mathbf{P}(\mathbf{x}) + (\mathbf{M}(\mathbf{x}) \otimes \mathcal{D}_{\mathbf{x}^\top}\{\mathbf{P}(\mathbf{x})\})\,(\mathbf{K}_{q,n} \otimes \mathbf{I}_v), \tag{C.7}$$

   where $\mathbf{K}_{q,n}$ is a $qn \times qn$ commutation matrix and I_v is an $v \times v$ identity matrix (see Magnus and Neudecker, 2019).

D Stochastic optimal control problem

Consider the following infinite horizon discounted stochastic optimal control problem

$$V(\mathbf{x}_0) = \max_{\mathbf{u}\in\mathbb{U}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \pi(\mathbf{x}, \mathbf{u}) \, \mathrm{d}t \right]$$

subject to

$$\mathrm{d}\mathbf{x} = \mathbf{b}(\mathbf{x}, \mathbf{u}; \eta) \, \mathrm{d}t + \sqrt{\eta} \boldsymbol{\sigma}(\mathbf{x}, \mathbf{u}) \, \mathrm{d}\mathbf{w}, \quad \mathbf{x}(0) = \mathbf{x}_0 \text{ given},$$

where $\mathbf{x} \in \mathbb{X}$ is the $n_x \times 1$ vector of state variables from the state space $\mathbb{X} \subseteq \mathbb{R}^{n_x}$, $\mathbf{u} \in \mathbb{U}$ is the $n_u \times 1$ vector of control variables from the control region $\mathbb{U} \subseteq \mathbb{R}^{n_u}$, $\mathbf{w}$ is an $n_w \times 1$ vector of mutually independent Brownian motions, and $\eta \in \mathbb{R}^+$ is a perturbation parameter rescaling the variance in the model. Further, $\pi : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is the reward function, $\rho \geq 0$ the discount rate, $\mathbf{b} : \mathbb{X} \times \mathbb{U} \times \mathbb{R}^+ \to \mathbb{R}^{n_x}$ the vector-valued drift function, and $\boldsymbol{\sigma} : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^{n_x \times n_w}$ is the diffusion matrix defining the $n_x \times n_x$ variance-covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$.

Choosing an admissible control $\mathbf{u} \in \mathbb{U}$ and defining $V(\mathbf{x}) : \mathbb{X} \to \mathbb{R}$ as the value function, the Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\rho V(\mathbf{x}) = \max_{\mathbf{u}\in\mathbb{U}} \left\{ \pi(\mathbf{x}, \mathbf{u}) + \tfrac{1}{\mathrm{d}t} \mathbb{E}_t \left[\mathrm{d}V(\mathbf{x}) \right] \right\}.$$

Using Itô's lemma, it follows that

$$\mathrm{d}V(\mathbf{x}) = \left(V_{\mathbf{x}}(\mathbf{x})^\top \mathbf{b}(\mathbf{x}, \mathbf{u}; \eta) + \tfrac{1}{2}\eta \mathrm{tr} \left[\boldsymbol{\sigma}(\mathbf{x}, \mathbf{u}) \boldsymbol{\sigma}(\mathbf{x}, \mathbf{u})^\top V_{\mathbf{xx}}(\mathbf{x}) \right] \right) \mathrm{d}t + \sqrt{\eta} V_{\mathbf{x}}(\mathbf{x})^\top \boldsymbol{\sigma}(\mathbf{x}, \mathbf{u}) \, \mathrm{d}\mathbf{w},$$

where tr denotes the trace of a matrix, $V_{\mathbf{x}}$ is the $n_x \times 1$ vector of first-order derivatives of the value function with respect to the state vector, and $V_{\mathbf{xx}}$ is the corresponding $n_x \times n_x$ matrix of second-order derivatives. Taking the expectation of the integral form and using the martingale property of Brownian motions yields

$$\mathbb{E}_t \left[\mathrm{d}V(\mathbf{x}) \right] = \left(V_{\mathbf{x}}(\mathbf{x})^\top \mathbf{b}(\mathbf{x}, \mathbf{u}; \eta) + \tfrac{1}{2}\eta \mathrm{tr} \left[\boldsymbol{\Sigma}(\mathbf{x}, \mathbf{u}) V_{\mathbf{xx}}(\mathbf{x}) \right] \right) \mathrm{d}t,$$

and the HJB equation becomes

$$\rho V(\mathbf{x}) = \max_{\mathbf{u}\in\mathcal{U}} \left\{ \pi(\mathbf{x}, \mathbf{u}) + V_{\mathbf{x}}(\mathbf{x})^\top \mathbf{b}(\mathbf{x}, \mathbf{u}; \eta) + \tfrac{1}{2}\eta \left(\mathrm{vec} \, \boldsymbol{\Sigma}(\mathbf{x}, \mathbf{u}) \right)^\top \mathrm{vec} \, V_{\mathbf{xx}}(\mathbf{x}) \right\},$$

where we used the relation between the trace of a matrix and the vec operator (see Magnus and Neudecker, 2019). The first-order conditions read

$$\pi_{\mathbf{u}}(\mathbf{x}, \mathbf{u}) + \mathbf{b}_{\mathbf{u}}(\mathbf{x}, \mathbf{u}; \eta)^\top V_{\mathbf{x}}(\mathbf{x}) + \tfrac{1}{2}\eta \mathcal{D}_{\mathbf{u}} \{ (\mathrm{vec} \, \boldsymbol{\Sigma}(\mathbf{x}, \mathbf{u}))^\top \} \mathrm{vec} \, V_{\mathbf{xx}} = \mathbf{0}, \tag{D.1}$$

making the control variables a function of the state and costate variables $\mathbf{u} = \mathcal{U}(\mathbf{x}, V_{\mathbf{x}}, \eta V_{\mathbf{xx}}) = \mathcal{U}(\mathbf{x})$, where $\pi_{\mathbf{u}}$ is $n_u \times 1$ and $\mathbf{b}_{\mathbf{u}}$ is $n_x \times n_u$. Then, the maximized HJB equation reads

$$\rho V(\mathbf{x}) = \pi(\mathbf{x}, \mathcal{U}(\mathbf{x})) + V_{\mathbf{x}}(\mathbf{x})^{\top} \mathbf{b}(\mathbf{x}, \mathcal{U}(\mathbf{x}); \eta) + \tfrac{1}{2}\eta \left(\operatorname{vec} \mathbf{\Sigma}(\mathbf{x}, \mathcal{U}(\mathbf{x}))\right)^{\top} \operatorname{vec} V_{\mathbf{xx}}(\mathbf{x}). \quad \text{(D.2)}$$

Equations (D.1) and (D.2) determine the unknown optimal value and control functions in the state space, i.e., $V(\mathbf{x})$ and $\mathcal{U}(\mathbf{x})$.

The costate variable follows from the first-order derivative of the maximized HJB equation (D.2) with respect to the state vector, i.e.,

$$\begin{aligned}
\rho V_{\mathbf{x}}(\mathbf{x}) \;=\; & \pi_{\mathbf{x}}(\mathbf{x}, \mathcal{U}(\mathbf{x})) + \mathcal{U}_{\mathbf{x}}(\mathbf{x})^{\top} \pi_{\mathbf{u}}(\mathbf{x}, \mathcal{U}(\mathbf{x})) \\
& + V_{\mathbf{xx}}(\mathbf{x}) \mathbf{b}(\mathbf{x}, \mathcal{U}(\mathbf{x}); \eta) + \left(\mathbf{I}_{n_x} \otimes V_{\mathbf{x}}(\mathbf{x})^{\top}\right) \operatorname{vec}\left(\mathbf{b}_{\mathbf{x}}(\mathbf{x}, \mathcal{U}(\mathbf{x}); \eta)\right) \\
& + \left(\mathbf{I}_{n_x} \otimes V_{\mathbf{x}}(\mathbf{x})^{\top}\right) \left(\mathcal{U}_{\mathbf{x}}(\mathbf{x}) \otimes \mathbf{I}_{n_x}\right) \operatorname{vec}\left(\mathbf{b}_{\mathbf{u}}(\mathbf{x}, \mathcal{U}(\mathbf{x}); \eta)\right) \\
& + \tfrac{1}{2}\eta \Big(\left(\mathbf{\Sigma}_{\mathbf{x}}(\mathbf{x}, \mathcal{U}(\mathbf{x})) + \mathcal{U}_{\mathbf{x}}(\mathbf{x})^{\top} \mathcal{D}_{\mathbf{u}^{\top}}\{(\operatorname{vec} \mathbf{\Sigma}(\mathbf{x}, \mathcal{U}(\mathbf{x})))^{\top}\}\right) \operatorname{vec} V_{\mathbf{xx}}(\mathbf{x}) \\
& + \left(\mathbf{I}_{n_x} \otimes (\operatorname{vec} \mathbf{\Sigma}(\mathbf{x}, \mathcal{U}(\mathbf{x})))^{\top}\right) \operatorname{vec} V_{\mathbf{xxx}}(\mathbf{x}) \Big).
\end{aligned}$$

Using the envelope theorem together with the properties of the vec operator and of the Kronecker product, the costate equation becomes

$$\begin{aligned}
\rho V_{\mathbf{x}}(\mathbf{x}) = \; & \pi_{\mathbf{x}}(\mathbf{x}, \mathcal{U}(\mathbf{x})) + \tfrac{1}{2}\eta \mathbf{\Sigma}_{\mathbf{x}}(\mathbf{x}, \mathcal{U}(\mathbf{x})) \operatorname{vec} V_{\mathbf{xx}}(\mathbf{x}) + \mathbf{b}_{\mathbf{x}}(\mathbf{x}, \mathcal{U}(\mathbf{x}); \eta)^{\top} V_{\mathbf{x}}(\mathbf{x}) \\
& + V_{\mathbf{xx}}(\mathbf{x}) \mathbf{b}(\mathbf{x}, \mathcal{U}(\mathbf{x}); \eta) + \tfrac{1}{2}\eta \left(\mathbf{I}_{n_x} \otimes (\operatorname{vec} \mathbf{\Sigma}(\mathbf{x}, \mathcal{U}(\mathbf{x})))^{\top}\right) \operatorname{vec} V_{\mathbf{xxx}}(\mathbf{x}),
\end{aligned}$$

which can be compactly written as

$$\mathbf{0} = \mathbf{a}(\mathbf{x}, V_{\mathbf{x}}, \eta V_{\mathbf{xx}}) + V_{\mathbf{xx}} \mathbf{b}(\mathbf{x}, V_{\mathbf{x}}, \eta V_{\mathbf{xx}}; \eta) + \eta V_{\mathbf{xxx}} \mathbf{c}(\mathbf{x}, V_{\mathbf{x}}, \eta V_{\mathbf{xx}}),$$

where

$$\begin{aligned}
\mathbf{a}(\mathbf{x}, V_{\mathbf{x}}, \eta V_{\mathbf{xx}}) &= \pi_{\mathbf{x}} + \tfrac{1}{2}\eta \mathbf{\Sigma}_{\mathbf{x}} \operatorname{vec} V_{\mathbf{xx}} + \mathbf{b}_{\mathbf{x}}^{\top} V_{\mathbf{x}} - \rho V_{\mathbf{x}} \\
\mathbf{c}(\mathbf{x}, V_{\mathbf{x}}, \eta V_{\mathbf{xx}}) &= \tfrac{1}{2} \operatorname{vec} \mathbf{\Sigma}.
\end{aligned}$$

E The stochastic growth model

E.1 The HJB equation and the first-order conditions

By letting $X_0 = b = 0$ and $\xi \to \infty$, the model in Section 2 collapses to

$$V(K_0, A_0; \eta) = \max_{\{C_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} \mathrm{d}t \right]$$

subject to

$$\begin{aligned} \mathrm{d}K_t &= (\exp(A_t) K_t^{\alpha} - C_t - \delta K_t)\, \mathrm{d}t, \quad K(0) = K_0 > 0 \text{ given}, \\ \mathrm{d}A_t &= -\rho_A A_t \mathrm{d}t + \sqrt{\eta \sigma_A^2} \mathrm{d}B_{A,t}, \quad A(0) = A_0 \text{ given}, \end{aligned}$$

where we have included explicitly the perturbation parameter, η, that rescales the amount of variance in the model.

The HJB equation to the planner's problem is

$$\rho V(K_t, A_t; \eta) = \max_{C_t \in \mathbb{R}^+} \left\{ \frac{C_t^{1-\gamma}}{1-\gamma} + \frac{1}{\mathrm{d}t} \mathbb{E}_t \Big[\mathrm{d}V(K_t, A_t; \eta) \Big] \right\}.$$

Using Itô's lemma together with the properties of stochastic integrals, the HJB equation becomes

$$\begin{aligned} \rho V(K_t, A_t; \eta) = \max_{C_t \in \mathbb{R}^+} \Bigg\{ & \frac{C_t^{1-\gamma}}{1-\gamma} + (\exp(A_t) K_t^{\alpha} - C_t - \delta K_t) V_K(K_t, A_t; \eta) \\ & - \rho_A A_t V_A(K_t, A_t; \eta) + \tfrac{1}{2} \eta \sigma_A^2 V_{AA}(K_t, A_t; \eta) \Bigg\}. \end{aligned}$$

The first-order condition for an interior solution is

$$C_t^{-\gamma} = V_K(K_t, A_t; \eta), \tag{E.1}$$

making optimal consumption a function of the state variables and the perturbation parameter, $C_t = C(K_t, A_t; \eta)$.

E.2 Competitive equilibrium and the Euler equation

Substituting the first-order condition yields the maximized HJB equation

$$\begin{aligned} \rho V(K_t, A_t; \eta) = \tfrac{C(K_t, A_t; \eta)^{1-\gamma}}{1-\gamma} & + (\exp(A_t) K_t^{\alpha} - C(K_t, A_t; \eta) - \delta K_t) V_K(K_t, A_t; \eta) \\ & - \rho_A A_t V_A(K_t, A_t; \eta) + \tfrac{1}{2} \eta \sigma_A^2 V_{AA}(K_t, A_t; \eta) \end{aligned} \tag{E.2}$$

from which we can obtain the costate variable for capital (using the envelope theorem) as

$$\begin{aligned}\rho V_K(K_t, A_t; \eta) &= (\exp(A_t)K_t^{\alpha} - C(K_t, A_t; \eta) - \delta K_t)V_{KK}(K_t, A_t; \eta) \\ &+ (\alpha \exp(A_t)K_t^{\alpha-1} - \delta)V_K(K_t, A_t; \eta) - \rho_A A_t V_{AK}(K_t, A_t; \eta) + \tfrac{1}{2}\eta\sigma_A^2 V_{AAK}(K_t, A_t; \eta).\end{aligned}$$

Using Itô's lemma, the evolution of the costate variable, V_K, is given by

$$\begin{aligned}\mathrm{d}V_K(K_t, A_t; \eta) &= V_{KK}(K_t, A_t; \eta)\mathrm{d}K_t + V_{KA}(K_t, A_t; \eta)\mathrm{d}A_t + \tfrac{1}{2}\eta\sigma_A^2 V_{KAA}(K_t, A_t; \eta)\mathrm{d}t \\ &= (\rho - \alpha \exp(A_t)K_t^{\alpha-1} + \delta)V_K(K_t, A_t; \eta)\mathrm{d}t + V_{KA}(K_t, A_t; \eta)\sqrt{\eta\sigma_A^2}\mathrm{d}B_{A,t},\end{aligned}$$

which describes the dynamics of the marginal utility of consumption. Using the first-order condition, we obtain

$$\mathrm{d}C_t^{-\gamma} = (\rho - \alpha \exp(A_t)K_t^{\alpha-1} + \delta)C_t^{-\gamma}\mathrm{d}t - \gamma C_t^{-\gamma-1}C_A\sqrt{\eta\sigma_A^2}\mathrm{d}B_{A,t}.$$

After some algebra, the equation above becomes

$$\frac{\mathrm{d}C_t}{C_t} = \left[\tfrac{1}{\gamma}\left(\alpha \exp(A_t)K_t^{\alpha-1} - \delta - \rho\right) + \tfrac{1}{2}(1+\gamma)\left(\tfrac{C_A}{C_t}\right)^2 \eta\sigma_A^2\right]\mathrm{d}t + \left(\tfrac{C_A}{C_t}\right)\sqrt{\eta\sigma_A^2}\mathrm{d}B_{A,t},$$

which is the *Euler equation for consumption* in (50) (see Posch, 2011).

The system of equations formed by (E.1) and (E.2) characterize the unknown functions $V(K, A; \eta)$ and $C(K, A; \eta)$ that define the general equilibrium in this economy. Alternatively, this equilibrium can be obtained from the system of PDEs for the costate variables, V_K and V_A, and the first-order condition for consumption.

Notice that since we are only interested in the optimal policy for consumption, the first-order condition suggests that it is enough to study the solution to system of non-linear functional equations in the unknown functions $\{V_K(K_t, A_t; \eta), C(K_t, A_t; \eta)\}$

$$\begin{aligned}0 &= (\exp(A_t)K_t^{\alpha} - C(K_t, A_t; \eta) - \delta K_t)V_{KK}(K_t, A_t; \eta) + (\alpha \exp(A_t)K_t^{\alpha-1} - \delta)V_K(K_t, A_t; \eta) \\ &\quad -\rho_A A_t V_{AK}(K_t, A_t; \eta) + \tfrac{1}{2}\eta\sigma_A^2 V_{AAK}(K_t, A_t; \eta) - \rho V_K(K_t, A_t; \eta), \\ 0 &= C(K_t, A_t; \eta)^{-\gamma} - V_K(K_t, A_t; \eta).\end{aligned}$$

Substitution of the first-order condition into the PDE for the costate variable yields (38) which can be compactly written as

$$\mathcal{H}\left(\mathbf{x}, \mathbf{y}, \mathbf{y}_{\mathbf{x}}, \mathbf{y}_{\mathbf{xx}}; \eta\right) := \mathbf{a}\left(\mathbf{x}, \mathbf{y}\right) + \mathbf{y}_{\mathbf{x}}\mathbf{b}\left(\mathbf{x}, \mathbf{y}\right) + \eta\mathbf{y}_{\mathbf{xx}}\mathbf{c}\left(\mathbf{x}, \mathbf{y}\right) = 0,$$

with $\mathbf{y} = V_K$ and $\mathbf{x} = [K_t, A_t]^{\top}$.

E.3 Approximate solution

We approximate the unknown policy function $V_K = V_K(K, A; \eta) = V_K(\mathbf{x}; \eta)$ by means of a Taylor series expansion around the DSS. The latter is given by the solution to (41). Substituting into the functional $\mathcal{H}$ yields the new functional equation

$$F(\mathbf{x}; \eta) := \mathcal{H}(\mathbf{x}, V_K(\mathbf{x}; \eta), V_{K\mathbf{x}}(\mathbf{x}; \eta), V_{K\mathbf{xx}}(\mathbf{x}; \eta); \eta) = \mathbf{0}.$$

A first-order perturbation approximation to $V_K(K, A; \eta)$ around the DSS is given by

$$V_K(K, A; \eta) = \overline{V_K} + \overline{V_{KK}}\left(K - \overline{K}\right) + \overline{V_{KA}}\left(A - \overline{A}\right) + \overline{V_{K\eta}}\eta,$$

where $\overline{V_K}$ is the DSS value of the costate variable. To find the unknown coefficients $\overline{V_{KK}}$ and $\overline{V_{KA}}$ differentiate $F(\mathbf{x}; \eta) = 0$ with respect to K and A to obtain

$$\begin{aligned}
F_K(K, A; \eta) &= (\alpha \exp(A) K^{\alpha-1} + \tfrac{1}{\gamma} V_K^{-1/\gamma-1} V_{KK} - \delta) V_{KK} \\
&\quad + (\exp(A) K^{\alpha} - V_K^{-1/\gamma} - \delta K) V_{KKK} \\
&\quad + \alpha(\alpha-1) \exp(A) K^{\alpha-2} V_K + (\alpha \exp(A) K^{\alpha-1} - \delta) V_{KK} \\
&\quad - \rho_A A V_{AKK} + \tfrac{1}{2} \eta \sigma_A^2 V_{AAKK} - \rho V_{KK} = 0, \\
F_A(K, A; \eta) &= (\exp(A) K^{\alpha} + \tfrac{1}{\gamma} V_K^{-1/\gamma-1} V_{KA}) V_{KK} + (\exp(A) K^{\alpha} - V_K^{-1/\gamma} - \delta K) V_{KKA} \\
&\quad + \alpha \exp(A) K^{\alpha-1} V_K + (\alpha \exp(A) K^{\alpha-1} - \delta) V_{KA} \\
&\quad - \rho_A V_{AK} - \rho_A A V_{AKA} + \tfrac{1}{2} \eta \sigma_A^2 V_{AAKA} - \rho V_{KA} = 0,
\end{aligned}$$

which evaluated at the DSS, $(K, A, \eta) = (\overline{K}, \overline{A}, 0)$, become a system of equations in the unknowns $\overline{V_{KK}}$ and $\overline{V_{KA}}$

$$\begin{aligned}
F_K\left(\overline{K}, \overline{A}; 0\right) &= (\rho + \tfrac{1}{\gamma} \overline{V_K}^{-1/\gamma-1} \overline{V_{KK}}) \overline{V_{KK}} + \alpha(\alpha-1) \overline{K}^{\alpha-2} \overline{V_K} = 0, \\
F_A\left(\overline{K}, \overline{A}; 0\right) &= (\overline{K}^{\alpha} + \tfrac{1}{\gamma} \overline{V_K}^{-1/\gamma-1} \overline{V_{KA}}) \overline{V_{KK}} + (\delta + \rho) \overline{V_K} - \rho_A \overline{V_{KA}} = 0.
\end{aligned}$$

Notice that the system formed by $F_K = F_A = 0$ is partially decoupled. Hence, we first solve for $\overline{V_{KK}}$ from the quadratic equation implied by $F_K\left(\overline{K}, \overline{A}; 0\right) = 0$

$$\tfrac{1}{\gamma} \overline{V_K}^{-1/\gamma-1} \overline{V_{KK}}^2 + \rho \overline{V_{KK}} + \alpha(\alpha-1) \overline{K}^{\alpha-2} \overline{V_K} = 0,$$

which yields

$$\overline{V_{KK}} = \frac{-\rho \pm \sqrt{\rho^2 - 4\tfrac{1}{\gamma}\alpha(\alpha-1) \overline{K}^{\alpha-2} \overline{V_K}^{-1/\gamma}}}{2\tfrac{1}{\gamma} \overline{V_K}^{-1/\gamma-1}}.$$

To ensure that the value function is strictly concave in capital, we choose the root implying

$\overline{V_{KK}} < 0$. Next, we compute $\overline{V_{KA}}$ from the linear equation formed by $F_A\left(\overline{K}, \overline{A}; 0\right) = 0$

$$\left(\overline{K}^{\alpha} + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KA}}\right)\overline{V_{KK}} + (\delta+\rho)\,\overline{V_K} - \rho_A\overline{V_{KA}} = 0,$$

which yields

$$\overline{V_{KA}} \;=\; -\left(\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KK}} - \rho_A\right)^{-1}\left(\overline{K}^{\alpha}\overline{V_{KK}} + (\delta+\rho)\overline{V_K}\right).$$

To obtain $\overline{V_{K\eta}}$ we differentiate $F(\mathbf{x};\eta)$ with respect to η and get

$$\begin{aligned} F_\eta(K,A;\eta) \;=\;& \tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{K\eta}V_{KK} + (\exp(A)K^{\alpha} - V_K^{-1/\gamma} - \delta K)V_{KK\eta} \\ &+(\alpha\exp(A)K^{\alpha-1} - \delta)V_{K\eta} - \rho_A A V_{AK\eta} \\ &+\tfrac{1}{2}\sigma_A^2 V_{AAK} + \tfrac{1}{2}\eta\sigma_A^2 V_{AAK\eta} - \rho V_{K\eta} = 0, \end{aligned}$$

which evaluated at the DSS becomes

$$F_\eta\left(\overline{K}, \overline{A}; 0\right) \;=\; \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{K\eta}V_{KK}} + \tfrac{1}{2}\sigma_A^2\overline{V_{AAK}} = 0.$$

This is a linear inhomogeneous equation in the unknown $\overline{V_{K\eta}}$ and its solution is given by

$$\overline{V_{K\eta}} \;=\; -\tfrac{1}{2}\sigma_A^2\frac{\overline{V_{AAK}}}{\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KK}}},$$

which depends linearly on $\overline{V_K}$, $\overline{V_{KK}}$, and $\overline{V_{AAK}}$. The dependence of $\overline{\mathbf{g}}_\eta$ on $\overline{\mathbf{g}}_{\mathbf{x}}$ and $\overline{\mathbf{g}}_{\mathbf{xx}}$ implies that a first-order approximation to the policy function $\mathbf{y} = \mathbf{g}(\mathbf{x};\eta)$ requires further differentiation of F with respect to $\mathbf{x}$ to be complete. Then, to find $\overline{\mathbf{g}}_{\mathbf{xx}}$ we compute

$$\begin{aligned} F_{KK}(K,A;\eta) \;=\;& \left(\alpha(\alpha-1)\exp(A)K^{\alpha-2}\left(\tfrac{1}{\gamma}+1\right)\tfrac{1}{\gamma}V_K^{-1/\gamma-2}V_{KK}^2 + \tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{KKK}\right)V_{KK} \\ &+2\left(\alpha\exp(A)K^{\alpha-1} + \tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{KK} - \delta\right)V_{KKK} \\ &+\left(\exp(A)K^{\alpha} - V_K^{-1/\gamma} - \delta K\right)V_{KKKK} + \alpha(\alpha-1)(\alpha-2)\exp(A)K^{\alpha-3}V_K \\ &+2\alpha(\alpha-1)\exp(A)K^{\alpha-2}V_{KK} + (\alpha\exp(A)K^{\alpha-1} - \delta - \rho)V_{KKK} \\ &-\rho_A A V_{AKKK} + \tfrac{1}{2}\eta\sigma_A^2 V_{AAKKK} = 0, \\ F_{KA}(K,A;\eta) \;=\;& \left(\alpha\exp(A)K^{\alpha-1} - \left(\tfrac{1}{\gamma}+1\right)\tfrac{1}{\gamma}V_K^{-1/\gamma-2}V_{KA}V_{KK} + \tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{KAA}\right)V_{KK} \\ &+\left(\alpha\exp(A)K^{\alpha-1} + \tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{KK} - \delta\right)V_{KKA} \\ &+\left(\exp(A)K^{\alpha} + \tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{KA}\right)V_{KKK} + \left(\exp(A)K^{\alpha} - V_K^{-1/\gamma} - \delta K\right)V_{KKKA} \\ &+\alpha(\alpha-1)\exp(A)K^{\alpha-2}V_K + \alpha(\alpha-1)\exp(A)K^{\alpha-2}V_{KA} + \alpha\exp(A)K^{\alpha-1}V_{KK} \\ &+(\alpha\exp(A)K^{\alpha-1} - \delta - \rho)V_{KKA} - \rho_A V_{AKK} - \rho_A A V_{AKKA} + \tfrac{1}{2}\eta\sigma_A^2 V_{AAKKA} = 0, \end{aligned}$$

$$
\begin{aligned}
F_{AA}\left(K,A;\eta\right) &= \left(\exp(A)K^{\alpha}-\left(\tfrac{1}{\gamma}+1\right)\tfrac{1}{\gamma}V_K^{-1/\gamma-2}V_{KA}^2+\tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{KAA}\right)V_{KK}\\
&\quad +2\left(\exp(A)K^{\alpha}+\tfrac{1}{\gamma}V_K^{-1/\gamma-1}V_{KA}\right)V_{KKA}\\
&\quad +\left(\exp(A)K^{\alpha}-V_K^{-1/\gamma}-\delta K\right)V_{KKAA}+\alpha\exp(A)K^{\alpha-1}V_K\\
&\quad +2\alpha\exp(A)K^{\alpha-1}V_{KA}+(\alpha\exp(A)K^{\alpha-1}-\delta-\rho)V_{KAA}\\
&\quad -2\rho_A V_{AKA}-\rho_A A V_{AKAA}+\tfrac{1}{2}\eta\sigma_A^2 V_{AAKAA}=0.
\end{aligned}
$$

When evaluated at the DSS, the system of equations formed by $F_{KK}=F_{KA}=F_{AA}=0$ reduces to the system of linear equations

$$
\begin{aligned}
F_{KK}\left(\overline{K},\overline{A};0\right) &= \left(\alpha(\alpha-1)\overline{K}^{\alpha-2}+\left(-\tfrac{1}{\gamma}-1\right)\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-2}\overline{V_{KK}}^2+\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KKK}}\right)\overline{V_{KK}}\\
&\quad +2\left(\rho+\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KK}}\right)\overline{V_{KKK}}\\
&\quad +\alpha(\alpha-1)(\alpha-2)\overline{K}^{\alpha-3}\overline{V_K}+2\alpha(\alpha-1)\overline{K}^{\alpha-2}\overline{V_{KK}}=0,\\
F_{KA}\left(\overline{K},\overline{A};0\right) &= \left(\alpha\overline{K}^{\alpha-1}+\left(-\tfrac{1}{\gamma}-1\right)\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-2}\overline{V_{KA}V_{KK}}+\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KAA}}\right)\overline{V_{KK}}\\
&\quad +\left(\rho+\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KK}}\right)\overline{V_{KKA}}+\left(\overline{K}^{\alpha}+\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KA}}\right)\overline{V_{KKK}}\\
&\quad +\alpha(\alpha-1)\overline{K}^{\alpha-2}\overline{V_K}+\alpha(\alpha-1)\overline{K}^{\alpha-2}\overline{V_{KA}}+\alpha\overline{K}^{\alpha-1}\overline{V_{KK}}-\rho_A\overline{V_{AKK}}=0,\\
F_{AA}\left(\overline{K},\overline{A};0\right) &= \left(\overline{K}^{\alpha}+\left(-\tfrac{1}{\gamma}-1\right)\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-2}\overline{V_{KA}}^2+\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KAA}}\right)\overline{V_{KK}}\\
&\quad +2\left(\overline{K}^{\alpha}+\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KA}}\right)\overline{V_{KKA}}\\
&\quad +\alpha\overline{K}^{\alpha-1}\overline{V_K}+2\alpha\overline{K}^{\alpha-1}\overline{V_{KA}}-2\rho_A\overline{V_{AKA}}-\rho\overline{V_{KAA}}=0,
\end{aligned}
$$

from which we recover $\overline{V_{AAK}}$ to compute $\overline{V_{K\eta}}$ and, hence, complete the first-order approximation to $V_K(K,A;\eta)$.

Using the approximated V_K, we obtain a first-order approximation to the consumption function by linearizing the first-order condition around the DSS. More specifically,

$$
\begin{aligned}
C(K,A,\eta) &= V_K(K,A,\eta)^{-1/\gamma}\\
&\approx \overline{C}-\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\left(V_K-\overline{V_K}\right)\\
&= \overline{C}-\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\left(\overline{V_{KK}}\left(K-\overline{K}\right)+\overline{V_{KA}}\left(A-\overline{A}\right)+\overline{V_{K\eta}}\eta\right)\\
&= \overline{C}+\overline{C_K}\left(K-\overline{K}\right)+\overline{C_A}\left(A-\overline{A}\right)+\overline{C_\eta}\eta,
\end{aligned}
$$

where $\overline{C}$ denotes the deterministic steady state for consumption and $\overline{C_K}$, $\overline{C_A}$, and $\overline{C_\eta}$ are given, respectively, by (47), (48), and (49).

E.4 Risky steady state

As shown in Section 3.4, the first-order approximation to the RSS value for the state variables, $(\widehat{K}, \widehat{A})$, is given as the solution to

$$\begin{bmatrix} \exp(\widehat{A})\widehat{K}^{\alpha} - \left(\overline{V_K} + \overline{V_{KK}}\left(\widehat{K} - \overline{K}\right) + \overline{V_{KA}}\left(\widehat{A} - \overline{A}\right) + \overline{V_{K\eta}}\right)^{-1/\gamma} - \delta\widehat{K} \\ -\rho_A\widehat{A} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

from where it follows that $\widehat{A} = 0$, and $\widehat{K}$ solves the nonlinear equation

$$\widehat{K}^{\alpha} - \left(\overline{V_K} + \overline{V_{KK}}\left(\widehat{K} - \overline{K}\right) + \overline{V_{K\eta}}\right)^{-1/\gamma} - \delta\widehat{K} = 0.$$

Substitution of $(\widehat{K}, \widehat{A})$ in (42) and (46) yields an approximation to the RSS values for $\widehat{V_K}$ and $\widehat{C}$, respectively.

Alternatively, an approximate closed-form solution to the first-order RSS can be obtained by first linearizing the drift of the capital stock accumulation equation around the DSS which yields

$$b(K, A, V_K; \eta) = \rho\left(K - \overline{K}\right) + \exp(\overline{A})\overline{K}^{\alpha}\left(A - \overline{A}\right) + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\left(V_K - \overline{V_K}\right).$$

Next, we substitute the first-order approximation to $V_K(K, A; \eta)$ to obtain

$$\begin{aligned} b(K, A; \eta) = &\left(\rho + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KK}}\right)\left(K - \overline{K}\right) \\ &+ \left(\exp(\overline{A})\overline{K}^{\alpha} + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KA}}\right)\left(A - \overline{A}\right) + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{K\eta}}\eta, \end{aligned}$$

which evaluated at the RSS becomes (using the fact that $\widehat{A} = \overline{A} = 0$)

$$b(\widehat{K}, \widehat{A}; 1) = \left(\rho + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KK}}\right)\left(\widehat{K} - \overline{K}\right) + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{K\eta}} = 0.$$

Then, the RSS value for the capital stock is given by

$$\widehat{K} = \overline{K} - \left(\rho + \tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{KK}}\right)^{-1}\tfrac{1}{\gamma}\overline{V_K}^{-1/\gamma-1}\overline{V_{K\eta}}.$$

Using (47)-(49) we can rewrite the above as

$$\begin{aligned} \widehat{K} &= \overline{K} + \left(\rho - \overline{C_K}\right)^{-1}\overline{C_\eta} \\ &= \overline{K} - \tfrac{1}{2}\sigma_A^2\frac{(1+\gamma)\overline{C}\left(\overline{C_A}/\overline{C}\right)^2 - \overline{C_{AA}}}{\left(\rho - \overline{C_K}\right)\overline{C_K}}, \end{aligned}$$

which shows the dependence of the RSS on the model's source of risk, σ_A. Moreover, concavity of the optimal consumption function implies that $\widehat{K} < \overline{K}$ as long as $\rho > \overline{C_K}$.

F The discrete-time model

This appendix introduces a discrete-time version of the prototype RBC model studied in the paper. The model follows closely that in Jermann (1998). Table F1 gives a summary of the model setup in continuous and discrete time. We also provide a summary of the perturbation method for discrete-time economies in the spirit of Schmitt-Grohe and Uribe (2004) and Fernández-Villaverde et al. (2016) and stress how certainty equivalence results from a first-order approximation. Finally, we discuss the concept of the risky steady state and how to approximate it based on the work by de Groot (2013).

F.1 The social planner's problem

Consider the problem faced by a social planner with preferences over streams of consumption, C_t, which are summarized by the expected present discounted value of a representative agent's life time utility with subjective discount factor $\beta \in (0,1)$

$$\tilde{U}_0 := \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{(C_t - X_t)^{1-\gamma}}{1-\gamma} \right], \tag{F.1}$$

We assume that consumption is a non-negative choice that cannot fall below a subsistence level, $C_t \geq X_t$, where X_t denotes internal habits in consumption. Following Grishchenko (2010), the household's internal habit is defined as $X_t = \tilde{b} \sum_{s=0}^{t-1} (1-\tilde{a})^{t-s-1} C_s$, or equivalently,

$$X_t = \tilde{b} C_{t-1} + (1-\tilde{a}) X_{t-1}. \tag{F.2}$$

The parameters $\tilde{a}$ and $\tilde{b}$ share the same interpretation as in the main text, although a tilde on top of the parameters indicates that their value might not be the same due to the discrete-time nature of the problem. Note that once again the household preferences collapse to the standard time-separable case if $X_0 = \tilde{b} = 0$.

The aggregate output of the economy is produced using the Cobb-Douglas technology

$$Y_t = \exp(A_t) K_t^{\alpha} L_t^{1-\alpha}, \tag{F.3}$$

where K_t is the aggregate capital stock, and L_t is the perfectly inelastic labor supply (normalized to one $\forall t \geq 0$). The former accumulates according to

$$K_{t+1} = \Phi(I_t/K_t) K_t + (1-\delta) K_t, \quad K_0 > 0, \tag{F.4}$$

where

$$\Phi(I_t/K_t) = \tfrac{a_1}{1-1/\xi} (I_t/K_t)^{1-1/\xi} + a_2, \tag{F.5}$$

represents capital adjustment costs. The process A_t describing the evolution of total

factor productivity (TFP), $\exp(A_t)$, is assumed to follow the AR(1) process

$$A_{t+1} = \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1}, \quad A_0 > 0, \tag{F.6}$$

where $\tilde{\rho}_A \in (0,1)$ measures the persistence of TFP, $\tilde{\sigma}_A > 0$ its volatility, and $\epsilon_{A,t} \sim \mathcal{N}(0,1)$ is a TFP shock. Finally, the economy satisfies the aggregate resource constraint

$$Y_t = C_t + I_t. \tag{F.7}$$

The problem faced by the social planner is that of choosing the time path for consumption that maximizes (F.1) subject to the dynamic constraints (F.2),(F.4), and (F.6), and the static constraints (F.3), (F.5), and (F.7):

$$\widetilde{V}(K_0, A_0, X_0) = \max_{\{C_t \geq X_t \in \mathbb{R}^+\}_{t=0}^{\infty}} \tilde{U}_0 \quad \text{s.t.} \quad (F.2) - (F.7), \tag{F.8}$$

in which $C_t \geq X_t \in \mathbb{R}^+$ denotes the control variable at time $t \in \mathbb{Z}$, and $\widetilde{V}(K_0, X_0, A_0)$ the value of the optimal plan (value function) from the perspective of time $t = 0$. For any $t \in \{0, 1, 2, \dots\}$, the *Bellman equation* is given by

$$\widetilde{V}(K_t, A_t, X_t) = \max_{C_t \geq X_t \in \mathbb{R}^+} \left\{ \tfrac{(C_t - X_t)^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}_t \widetilde{V}(K_{t+1}, A_{t+1}, X_{t+1}) \right\} \tag{F.9}$$

subject to

$$\begin{aligned} K_{t+1} &= \Phi\left(\tfrac{\exp(A_t)K_t^{\alpha} - C_t}{K_t}\right) K_t + (1-\delta) K_t, \\ X_{t+1} &= \tilde{b} C_t + (1 - \tilde{a}) X_t, \\ A_{t+1} &= \tilde{\rho}_A A_t + \tilde{\sigma}_A \epsilon_{A,t+1}. \end{aligned}$$

The first-order condition for an interior solution is

$$(C_t - X_t)^{-\gamma} + \tilde{b}\beta \mathbb{E}_t \left[\widetilde{V}_{X,t+1}\right] = \Phi'\left(\tfrac{\exp(A_t)K_t^{\alpha} - C_t}{K_t}\right) \beta \mathbb{E}_t \left[\widetilde{V}_{K,t+1}\right], \tag{F.10}$$

where $\widetilde{V}_{K,t+1} := \widetilde{V}_K(K_{t+1}, X_{t+1}, A_{t+1})$ and $\widetilde{V}_{X,t+1} := \widetilde{V}_X(K_{t+1}, X_{t+1}, A_{t+1})$ are the partial derivatives of the value function with respect to the states K and X. Equation (F.10) makes optimal consumption a function of the state variables, $C_t^{\star} = C(K_t, A_t, X_t)$.

By means of the envelope theorem, the costate variable with respect to capital is defined by

$$\begin{aligned} \widetilde{V}_{K,t} = \beta \Bigg(\Phi'\left(\tfrac{\exp(A_t)K_t^{\alpha} - C_t}{K_t}\right) \left((\alpha - 1)\exp(A_t) K_t^{\alpha - 1} + \tfrac{C_t}{K_t} \right) \\ + \Phi\left(\tfrac{\exp(A_t)K_t^{\alpha} - C_t}{K_t}\right) + 1 - \delta \Bigg) \mathbb{E}_t \left[\widetilde{V}_{K,t+1}\right], \end{aligned}$$

while with respect to the habit by

$$\widetilde{V}_{X,t} = -\left(C_t - X_t\right)^{-\gamma} + \left(1-\tilde{a}\right)\beta\mathbb{E}_t\left[\widetilde{V}_{X,t+1}\right].$$

A solution to the planner's problem is given by the sequence $\left\{\widetilde{V}_{K,t}, \widetilde{V}_{X,t}, K_t, X_t, A_t\right\}_{t=0}^{\infty}$ that solves the boundary value problem (with appropriate transversality conditions) characterized by the system of equilibrium stochastic difference equations

$$\begin{aligned}
\widetilde{V}_{K,t} &= \beta\left(\Phi'\left(\tfrac{\exp(A_t)K_t^{\alpha}-C_t}{K_t}\right)\left((\alpha-1)\exp(A_t)K_t^{\alpha-1}+\tfrac{C_t}{K_t}\right)\right. \\
&\quad \left.+\,\Phi\left(\tfrac{\exp(A_t)K_t^{\alpha}-C_t}{K_t}\right)+1-\delta\right)\mathbb{E}_t\left[\widetilde{V}_{K,t+1}\right], && \text{(F.11)}\\
\widetilde{V}_{X,t} &= -\left(C_t - X_t\right)^{-\gamma} + \left(1-\tilde{a}\right)\beta\mathbb{E}_t\left[\widetilde{V}_{X,t+1}\right], && \text{(F.12)}\\
X_{t+1} &= \tilde{b}C_t + (1-\tilde{a})X_t, && \text{(F.13)}\\
K_{t+1} &= \Phi\left(\left(\exp(A_t)K_t^{\alpha} - C_t\right)/K_t\right)K_t + (1-\delta)K_t, && \text{(F.14)}\\
A_{t+1} &= \tilde{\rho}_A A_t + \tilde{\sigma}_A\epsilon_{A,t+1}, && \text{(F.15)}
\end{aligned}$$

together with initial conditions $K(0) = K_0$, $X(0) = X_0$, and $A(0) = A_0$, and where C_t solves the nonlinear algebraic equation in (F.10).

Table F1 gives a summary of the model setup in continuous and discrete time.

	Continuous time	Discrete time
Objective function	$\mathbb{E}_0\left[\int_0^{\infty} e^{-\rho t}\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\mathrm{d}t\right]$	$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\frac{(C_t-X_t)^{1-\gamma}}{1-\gamma}\right]$
Market clearing	$\exp(A_t)K_t^{\alpha}L^{1-\alpha} = C_t + I_t$	$\exp(A_t)K_t^{\alpha}L^{1-\alpha} = C_t + I_t$
Capital dynamics	$\mathrm{d}K_t = \left(\Phi\left(\frac{I_t}{K_t}\right)-\delta\right)K_t\mathrm{d}t$	$K_{t+1} = \left(\Phi\left(\frac{I_t}{K_t}\right)+(1-\delta)\right)K_t$
Habit dynamics	$\mathrm{d}X_t = (bC_t - aX_t)\,\mathrm{d}t$	$X_{t+1} = \tilde{b}C_{t-1} + \tilde{a}X_{t-1}$
TFP dynamics	$\mathrm{d}A_t = -\rho_A A_t\mathrm{d}t + \sigma_A\mathrm{d}B_{A,t}$	$A_{t+1} = \tilde{\rho}_A A_t + \tilde{\sigma}_A\epsilon_{A,t+1}$
TFP shock	$(B_{A,t+\Delta} - B_{A,t}) \sim N(0,\Delta)$	$\epsilon_{A,t} \sim N(0,1)$

Table F1. Summary of the two modeling frameworks.

F.2 Deterministic steady state

In the absence of uncertainty ($\tilde{\sigma}_A = 0$), the deterministic steady state is defined as an equilibrium in which all variables in the economy are constant. Hence, given the

assumptions on the capital adjustment cost function in (F.5), the deterministic steady state is fully characterized by

$$\begin{aligned}
\overline{A} &= 0, && \text{(F.16)}\\
\overline{K} &= \left[\tfrac{\alpha \exp(\overline{A})}{\rho+\delta}\right]^{\frac{1}{1-\alpha}}, && \text{(F.17)}\\
\overline{C} &= \exp(\overline{A})\overline{K}^{\alpha} - \delta\overline{K}, && \text{(F.18)}\\
\overline{X} &= \tfrac{b}{a}\overline{C}, && \text{(F.19)}\\
\beta\overline{\widetilde{V}_X} &= -\tfrac{1}{\rho+a}\left(\overline{C}-\overline{X}\right)^{-\gamma}, && \text{(F.20)}\\
\beta\overline{\widetilde{V}_K} &= \left(1-\tfrac{b}{\rho+a}\right)\left(\overline{C}-\overline{X}\right)^{-\gamma}, && \text{(F.21)}
\end{aligned}$$

where $\overline{\widetilde{V}_X}$ and $\overline{\widetilde{V}_K}$ denote the deterministic steady-state values of the costate variables for the capital stock and the habit formation. By setting $\beta = 1/(1+\rho)$, and $\tilde{b} = b$ and $\tilde{a} = a$, we ensure that the steady state values of the capital stock and the long-run habit-to-consumption ratio are equal in the discrete- and continuous-time models.

F.3 Perturbation method

The equilibrium conditions of the model are summarized by equations (F.11)−(F.15). As in the continuous time case, the policy functions that solve these conditions are not available in closed form and, therefore, will be approximated using perturbation methods. Let the augmented stochastic process driving TFP be given by

$$A_{t+1} = \tilde{\rho}_A A_t + \eta\tilde{\sigma}_A\epsilon_{A,t+1},$$

where η is the perturbation parameter that controls the standard deviation of TFP shocks (but not the variance as in the continuous-time model).

Following Schmitt-Grohe and Uribe (2004), the equilibrium conditions can be compactly written as

$$\mathbb{E}_t\left[\mathcal{H}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{x}_t; \eta)\right] = \mathbf{0}, \qquad \text{(F.22)}$$

where $\mathbf{x}_t = [K_t, X_t, A_t]^\top$ is the vector of state variables at time t, with initial value $\mathbf{x}_0$, $\mathbf{y}_t = \left[\widetilde{V}_{K,t}, \widetilde{V}_{X,t}, \widetilde{V}_{A,t}, C_t\right]^\top$ is the vector of control variables at time t, and $\mathcal{H}$ is an operator that collects the equilibrium conditions (F.11)−(F.15). The deterministic steady state is then defined as the pair $(\overline{\mathbf{y}}, \overline{\mathbf{x}})$ that solves

$$\mathcal{H}(\overline{\mathbf{y}}, \overline{\mathbf{y}}, \overline{\mathbf{x}}, \overline{\mathbf{x}}; 0) = \mathbf{0}. \qquad \text{(F.23)}$$

The solution to the discrete-time model in (F.22) takes the form

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t;\eta), \tag{F.24}$$

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t;\eta) + \eta\tilde{\sigma}_A\epsilon_{A,t+1}, \tag{F.25}$$

where $\mathbf{g}(\cdot)$ is a vector of unknown policy functions that maps every possible value of $\mathbf{x}_t$ into $\mathbf{y}_t$, and $\mathbf{h}(\cdot)$ is a vector of unknown policy functions that maps every possible value of $\mathbf{x}_t$ into $\mathbf{x}_{t+1}$. Substituting into the functional operator that defines the equilibrium delivers the new operator

$$F(\mathbf{x}_t;\eta) := \mathbb{E}_t\left[\mathcal{H}\left(\mathbf{g}(\mathbf{h}(\mathbf{x}_t;\eta)+\eta\tilde{\sigma}_A\epsilon_{A,t+1};\eta),\mathbf{g}(\mathbf{x}_t;\eta),\mathbf{h}(\mathbf{x}_t;\eta)+\eta\tilde{\sigma}_A\epsilon_{A,t+1},\mathbf{x}_t;\eta\right)\right] = \mathbf{0}. \tag{F.26}$$

A perturbation-based approximation to the solution of the problem (F.22) builds a Taylor series expansion of the unknown policy functions around the deterministic steady state using the fact that (F.26) holds for any values of $\mathbf{x}_t$ and η. The latter implies that all partial derivatives of the functional $F(\mathbf{x}_t;\eta)$ must be zero, i.e.,

$$F_{x_i^k\eta^j}(\mathbf{x}_t;\eta) = 0, \quad \forall x,\eta,i,k,j,$$

where $F_{x_i^k\eta^j}(\mathbf{x}_t;\eta)$ denotes the derivative of F with respect to the i-th element in $\mathbf{x}_t$ taken k times, and with respect to η taken j times evaluated at $(\mathbf{x}_t;\eta)$.

A first-order approximation to the policy functions is defined by

$$\begin{aligned}\mathbf{g}(\mathbf{x}_t;\eta) &= \mathbf{g}(\overline{\mathbf{x}};0)+\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{g}_{\eta}(\overline{\mathbf{x}};0)\eta,\\ \mathbf{h}(\mathbf{x}_t;\eta) &= \mathbf{h}(\overline{\mathbf{x}};0)+\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t-\overline{\mathbf{x}})+\mathbf{h}_{\eta}(\overline{\mathbf{x}};0)\eta,\end{aligned}$$

where $\mathbf{g}(\overline{\mathbf{x}};0)$ and $\mathbf{h}(\overline{\mathbf{x}};0)$ correspond to the deterministic steady-state values of the control and state variables derived from (F.23), and where the constants $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$, $\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)$, $\mathbf{g}_{\eta}(\overline{\mathbf{x}};0)$, $\mathbf{h}_{\eta}(\overline{\mathbf{x}};0)$ can be determined by solving the system of equations formed by

$$\begin{aligned}F_{x_i}(\overline{\mathbf{x}};0) &= 0 \quad \forall i,\\ F_{\eta}(\overline{\mathbf{x}};0) &= 0.\end{aligned}$$

We refer to the first set of equations (those not involving the perturbation parameter) as the perfect-foresight component of the approximation, and to the second set of equations as the stochastic component of it (cf. Andreasen and Kronborg, 2018). The system of equations resulting from the perfect-foresight component is quadratic in the unknowns $\mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)$ and $\mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)$. We pick the solution ensuring stability of the model's endogenous variables, i.e., the stable manifold (cf. Blanchard and Kahn, 1980; Klein, 2000). The remaining constants, $\mathbf{g}_{\eta}(\overline{\mathbf{x}};0)$ and $\mathbf{h}_{\eta}(\overline{\mathbf{x}};0)$, correspond to the solution of the system of equa-

tions formed by the stochastic component, the unique solution being $\mathbf{g}_\eta(\overline{\mathbf{x}};0) = \mathbf{h}_\eta(\overline{\mathbf{x}};0) = 0$ (cf. Schmitt-Grohe and Uribe, 2004; Fernández-Villaverde et al., 2016). Hence

$$\mathbf{g}(\mathbf{x}_t;\eta) = \mathbf{g}(\overline{\mathbf{x}};0) + \mathbf{g}_\mathbf{x}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}), \tag{F.27}$$

$$\mathbf{h}(\mathbf{x}_t;\eta) = \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_\mathbf{x}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}), \tag{F.28}$$

implying that up to a first order, the approximation exhibits certainty equivalence, i.e., the solution of the model is identical to the solution of the same model in the absence of uncertainty, $\eta = 0$.

Similarly, a second-order approximation to the policy functions is defined as

$$\begin{aligned}\mathbf{g}(\mathbf{x}_t;\eta) = \mathbf{g}(\overline{\mathbf{x}};0) + \mathbf{g}_\mathbf{x}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \mathbf{g}_\eta(\overline{\mathbf{x}};0)\eta \\ + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes \eta + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)\eta^2,\end{aligned}$$

and

$$\begin{aligned}\mathbf{h}(\mathbf{x}_t;\eta) = \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_\mathbf{x}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \mathbf{h}_\eta(\overline{\mathbf{x}};0)\eta \\ + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \mathbf{h}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes \eta + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)\eta^2,\end{aligned}$$

where the definition of the matrices $\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$, $\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)$, $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$, and $\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)$ can be found in Binning (2013a). These unknown coefficients correspond to the solution of the system of equations formed by

$$\begin{aligned}F_{x_i x_j}(\overline{\mathbf{x}};0) &= 0 \quad \forall i,j, \\ F_{\eta\eta}(\overline{\mathbf{x}};0) &= 0.\end{aligned}$$

As shown in Schmitt-Grohe and Uribe (2004), the cross derivatives $\mathbf{g}_{\mathbf{x}\eta}$ and $\mathbf{h}_{\mathbf{x}\eta}$ evaluated at $(\overline{\mathbf{x}};0)$ are zero, and hence the second-order perturbation reduces to

$$\mathbf{g}(\mathbf{x}_t;\eta) = \mathbf{g}(\overline{\mathbf{x}};0) + \mathbf{g}_\mathbf{x}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0), \tag{F.29}$$

$$\mathbf{h}(\mathbf{x}_t;\eta) = \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_\mathbf{x}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0). \tag{F.30}$$

Hence, solving a second-order approximation introduces a constant correction in the policy functions that account for the effects of risk given by $\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0)$ and $\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0)$, while the slopes of the policy functions are not affected by risk as $\mathbf{g}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0) = \mathbf{h}_{\mathbf{x}\eta}(\overline{\mathbf{x}};0) = 0$.

F.4 Calibration

For the numerical exercises presented in the paper we calibrate the discrete-time model as we do in the continuous time case. In particular, we set the risk aversion parameter and the share of capital income to $\gamma = 2$ and $\alpha = 0.36$, respectively. The annual values for the subjective discount rate and the depreciation rate are fixed to $\beta = 1/(1+\rho) = 0.9606$ and $\delta = 0.0963$, respectively. For the habit process we use $a = 1$ and $b = 0.82$, while the adjustment cost parameter is calibrated to $\xi = 0.3261$. Finally, following Christensen et al. (2016), the annual values for the persistence and volatility of the TFP are set to $\tilde{\rho}_A = 0.8145$ and $\tilde{\sigma}_A = 0.0278$, respectively.

F.5 Risky steady state

Following de Groot (2013), it is possible to approximate the risky steady state of a discrete-time economy by making use of the second-order approximation around the deterministic steady state. First, consider the second-order approximation to the transition equation for the state variables in (F.30)

$$\mathbf{x}_{t+1} = \mathbf{h}(\overline{\mathbf{x}};0) + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\mathbf{x}_t - \overline{\mathbf{x}}) \otimes (\mathbf{x}_t - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0) + \tilde{\sigma}_A \epsilon_{A,t+1}.$$

By setting the random disturbances to zero, $\epsilon_{A,t+1} = 0$, we compute the risky steady-state value of the state variables as the vector $\widehat{\mathbf{x}}$ that satisfies $\mathbf{x}_{t+1} = \mathbf{x}_t = \widehat{\mathbf{x}}$, and thus solves the quadratic equation

$$\widehat{\mathbf{x}} = \overline{\mathbf{x}} + \mathbf{h}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\widehat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\widehat{\mathbf{x}} - \overline{\mathbf{x}}) \otimes (\widehat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{h}_{\eta\eta}(\overline{\mathbf{x}};0).$$

Once $\widehat{\mathbf{x}}$ is computed, it is possible to back out the implied risky steady-state values for the control variables, $\widehat{\mathbf{y}}$, by simply inserting $\widehat{\mathbf{x}}$ into (F.29)

$$\widehat{\mathbf{y}} = \overline{\mathbf{y}} + \mathbf{g}_{\mathbf{x}}(\overline{\mathbf{x}};0)(\widehat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\mathbf{xx}}(\overline{\mathbf{x}};0)(\widehat{\mathbf{x}} - \overline{\mathbf{x}}) \otimes (\widehat{\mathbf{x}} - \overline{\mathbf{x}}) + \tfrac{1}{2}\mathbf{g}_{\eta\eta}(\overline{\mathbf{x}};0).$$

The corresponding risky steady state values for habit, capital stock, and consumption resulting from the calibration in Section F.4 are $\widehat{X} = 1.0608$, $\widehat{K} = 4.7184$, and $\widehat{C} = 1.2936$, respectively.

G Policy and Impulse-Response functions

For comparison purposes, this appendix reports the policy and impulse-response functions (IRF) obtained from the discrete-time model in Appendix F. They are computed using the software platform `Dynare`.

Figure G1 compares approximated policy functions for consumption across orders of approximation; on the left-hand side (LHS) for the continuous time case and on the right-hand side (RHS) for the discrete time case. Note that our calibration implies identical DSS across time assumptions. While the discrete-time first-order policy function for consumption (solid line, RHS) goes through the DSS (approximation point) due to certainty equivalence, the continuous-time First-Order approximation (solid line, LHS) that breaks certainty equivalence does not. Only by shutting down the risk-correction, the continuous-time First-Order (CE) approximation (dotted line, LHS) will go through the DSS. Hence, as claimed in the main text, in this respect the First-Order (CE) resembles the first-order approximation in discrete time. Further, note that in continuous time the First-Order approximation is close to the Second-Order approximation (dashed line, LHS) in the neighborhood of the DSS, while in discrete time this is not the case.

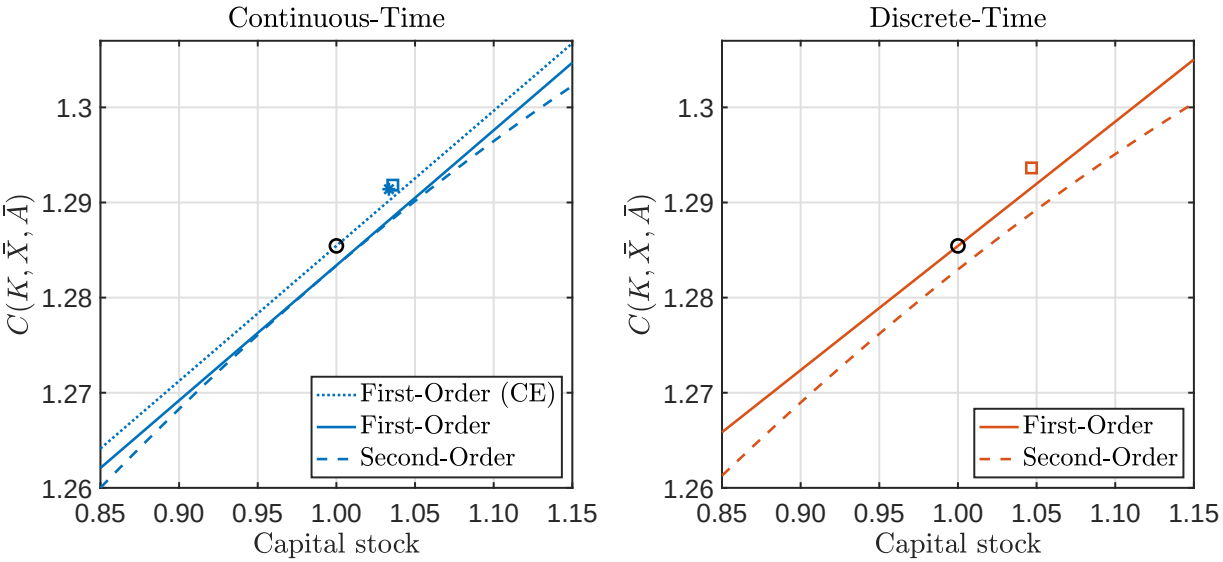

Figure G1. Continuous- and discrete-time approximated policy functions. First- and second-order approximations of the policy function for consumption around the DSS along the capital lattice while keeping habit and productivity at their DSS values, $C\left(K, \overline{X}, \overline{A}\right)$. A circle denotes the DSS, a star the first-order approximation and a square the second-order approximation of the RSS.

Figure G2 plots the approximated IRFs for consumption to a one standard deviation[1] shock in TFP across orders of approximation: on the LHS the continuous time case and on the RHS the discrete time case. As the first-order approximation in discrete time (solid line, RHS) is certainty equivalent, the corresponding IRF starts in the DSS, where it also converges to. A similar shape is observed for the IRF from the First-Order (CE) approximation in continuous time (dotted line, LHS). Again, in discrete time we observe a large difference between first- and second-order approximated IRFs (solid vs. dashed line, RHS), since only the second-order approximation is risk-sensitive. In contrast, the differences between the IRFs resulting from the First- and Second-Order approximation in continuous time (solid vs. dashed line, LHS) are minor reflecting the fact that both approximations of the policy function are similar in the neighborhood of the DSS (see Figure G1).

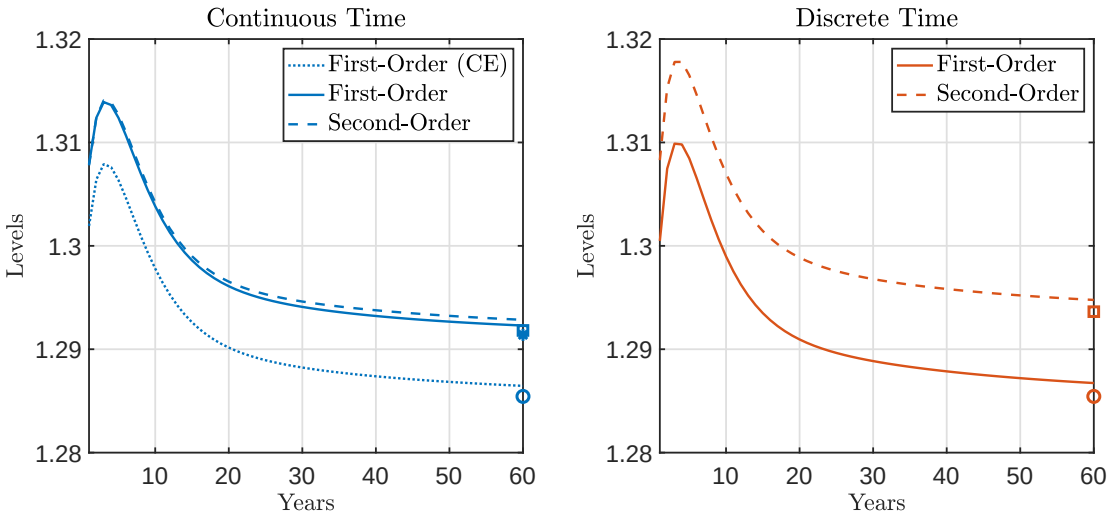

Figure G2. Comparison of IRFs to a shock in TFP. It plots the impulse response functions (IRFs) for the first- and second-order approximations of the policy function for consumption in continuous and discrete time. The variables in the economy are assumed to be in their corresponding DSS or RSS before the shock hits. A circle denotes the DSS, a star the first-order approximation and a square the second-order approximation of the RSS.

[1]More precisely, for ease of comparison, we impose in both time assumptions an impulse of one standard deviation of the continuous-time model, i.e., $\sigma_A = 0.0307$.

H Additional results

H.1 Pricing errors

Figure H1 reports the percentage (absolute) pricing errors for different approximations under the assumption that the true data generating process is given by the global approximation to the nonlinear stochastic model. The First-Order (CE), First-Order and Second-Order have been already introduced in Figure 3 in the main text.

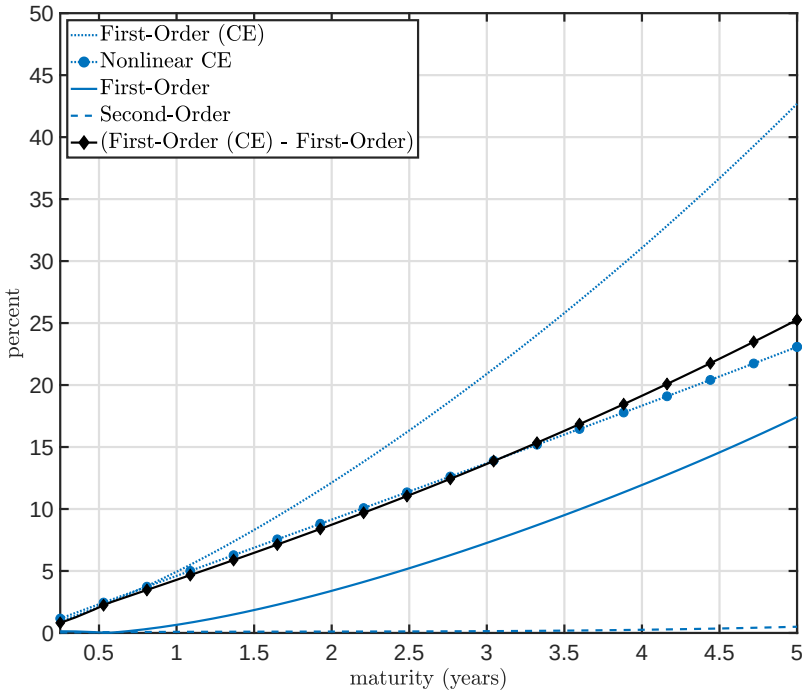

Figure H1. Decomposition of pricing errors. The graph plots the pricing errors resulting from First-Order (CE), First-Order, Nonlinear (CE), Second-Order, and the difference between the first two assuming that the true data generating process is the nonlinear stochastic solution.

The pricing error generated by the First-Order (CE) can be decomposed into: (i) the error stemming from the linearization of the nonlinear and stochastic policy function, which is captured by the First-Order approximation, and (ii) the error stemming from the imposition of certainty equivalence in the linear world. The latter is captured by the difference between First-Order (CE) and First-Order, and it is represented by the black line with diamonds: it measures the fraction of the pricing error that can be attributed to the imposition of certainty equivalence when using the First-Order (CE) solution. This measure can alternatively be interpreted as the reduction in the pricing errors that will be induced by the use of the (risk-sensitive) First-Order approximation.

Figure H1 presents an additional breakdown of the pricing errors generated by the use of the First-Order (CE). In particular, it is possible to decompose this error into (i) the error stemming from imposing certainty equivalence in the nonlinear world, and (ii) the error stemming from linearization in the presence of certainty equivalence. The

former is given by the approximation of the policy function using a global method in a deterministic environment (Nonlinear CE, blue line with circles), while the latter would be given by the difference between the Nonlinear CE and the First-Order (CE).

By comparing the black line with diamonds and the blue line with circles, we can infer the effects from imposing certainty equivalence on the quality of the approximation. The first one provides a measure of this error in the linearized world, while the second one does it in the nonlinear world. The results suggest that the error reduction one would obtain from using the First-Order approximation is very close to the error one makes when imposing certainty equivalence in the nonlinear global solution. This can be interpreted as our First-Order approximation removing all of the error stemming from certainty equivalence such that all the remaining error can be attributed to linearization and, thus, is inevitable. Therefore, the First-Order approximation in continuous time makes it possible to account for the effects of risk in a linear framework.

H.2 HJB equation errors

Following Judd and Guu (1993) and Parra-Alvarez (2018), we compute the unit-free HJB equation errors (HJB residuals) to assess the numerical accuracy of the approximation obtained by the perturbation method. This approach, which is similar to the Euler equation errors commonly reported in the discrete-time literature (see e.g. Aruoba et al., 2006), measures how much the HJB equation deviates from zero at a predefined set of values for the state variables. Therefore, the HJB equation errors can be regarded as a measure of optimization errors that indicate the welfare loss from suboptimal behavior due to numerical approximation errors.

For any admissible value of the state variables, the HJB residual measures how close to zero the HJB equation is when using method m to approximate the unknown policy functions. For the prototype economy of Section 2, the normalized or unit-free HJB residual is defined as

$$\begin{aligned} R^{(m)}_{\text{HJB}}(K,X,A) \;=\; & \left(\rho\bar{V}\right)^{-1}\left[\frac{(C^{(m)}(K,X,A)-X)^{1-\gamma}}{1-\gamma}\right. \\ & +(\Phi((\exp(A)K^{\alpha}-C^{(m)}(K,X,A))/K)-\delta)KV^{(m)}_{K}(K,X,A) \\ & +(bC^{(m)}(K,X,A)-aX)V^{(m)}_{X}(K,X,A)-\rho_{A}AV^{(m)}_{A}(K,X,A) \\ & \left.+\tfrac{1}{2}\sigma^{2}_{A}V^{(m)}_{AA}(K,X,A)-\rho V^{(m)}(K,X,A)\right], \end{aligned} \tag{H.1}$$

for all (K,X,A). Notice that $R_{\text{HJB}}(K,X,A)=0$ for all (K,X,A) when evaluated at the true policy functions.

Table H1 summarizes the HJB residuals for the different approximations to the un-

Approximation (m)	Average HJB residual	Max HJB residual
First-Order (CE)	-2.9737	-0.8747
First-Order (CE, LL)	-2.7101	-0.8274
First-Order	-2.8809	-0.8926
First-Order (LL)	-2.9969	-0.8396
Second-Order	-3.6630	-1.1846
Collocation	-4.7526	-3.4950

Table H1. HJB equation errors. The table reports the average HJB residuals, E_1, and the maximum HJB residual, E_∞, for each of approximation method m over the hypercube formed by $[0.85\bar{K}, 1.15\bar{K}] \times [0.85\bar{X}, 1.15\bar{X}] \times [0.9e^{\bar{A}}, 1.1e^{\bar{A}}]$.

known policy functions $C(K, X, A)$, $V_K(K, X, A)$, $V_X(K, X, A)$, $V_A(K, X, A)$. It reports the log10 magnitude of the following measures:

$$E_1^{(m)} = \frac{1}{n_K n_X n_A} \sum_{j=1}^{n_K} \sum_{i=1}^{n_X} \sum_{k=1}^{n_A} R_{\text{HJB}}^{(m)}(K_j, X_i, A_k), \quad \text{and} \quad E_\infty^{(m)} = \max_{i,j,k} R_{\text{HJB}}^{(m)}(K_j, X_i, A_k),$$

where $E_1^{(m)}$ is the average HJB residual, and $E_\infty^{(m)}$ is the maximum HJB residual, computed over a predefined state-space with n_K points in the capital stock lattice, n_X points in the habit lattice, and n_A points in the TFP lattice. Sometimes, $E_\infty^{(m)}$ is preferred since it bounds the error made by using a given approximation (cf. Aruoba et al., 2006). A value of -3 indicates that for every thousand units of life-time utils, the agent makes an error of 1 unit of life-time utils by using a given approximation.

The results show that the collocation method produces lower HJB residuals than those generated by (local) perturbation methods. Besides the approximations discussed in the main text, we also report, under the headings *First-Order (CE,LL)* and *First-Order (LL)*, the HJB residuals obtained when computing the first-order perturbation approximation on the logs of the unknown functions instead of their levels. They produce HJB residuals that are of the same order of magnitude of those produced when approximating the levels. Finally, while $\{V(K, X, A), C(K, X, A)\}$ in (12) provide necessary and sufficient conditions for an optimum, the functions $\{V_K(K, X, A), V_X(K, X, A), V_A(K, X, A), C(K, X, A)\}$ for the system (13) to (16) only provide necessary conditions. However, the small HJB residuals generated by all the approximations indicate that the solution also satisfy both the HJB equation and the first-order conditions.

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