Solving the New Keynesian Model in Continuous Time*

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Abstract

We show how to formulate and solve a New Keynesian model in continuous time. In our economy, monopolistic firms engage in infrequent price setting á la Calvo. We introduce shocks to preferences, to factor productivity, to monetary policy and to government expenditure, and show how the equilibrium system can be written in terms of 8 state variables. Our nonlinear and global numerical solution method allows us to compute equilibrium dynamics and impulse response functions in the time space, the collocation method based on Chebychev polynomials is used to compute the recursive-competitive equilibrium based on the continuous-time HJB equation in the policy function space. We illustrate advantages of continuous time by studying the effects on the zero lower bound of interest rates.

Keywords: Continuous-time DSGE models, Calvo price setting
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1. Introduction

New Keynesian (NK) models of the business cycle have become a fundamental tool in the study of aggregate fluctuations and in the design of monetary and fiscal policies. They fill the pages of journals and they are extensively used by central banks all around the world to assess the effects of different monetary interventions.

Nearly all of this extensive literature has worked with a formulation of the model in discrete time. This was in part because of the familiarity of macroeconomists with discrete time forms of previous models of the business cycle, as the real business cycle model, and in part because of the natural mapping of discrete time models with data, which come by construction in discrete observations.

There are, however, reasons to develop an alternative formulation of the model in continuous time. Dynamic equilibrium models written in continuous time can take advantage of a powerful set of mathematical tools developed in the fields of stochastic processes, optimal control, and PDEs. Thanks to these tools, many issues, such as adjustment costs, kinks, or other significant non-linearities can be easily handled and, often, we can even find closed-form solutions. This is particularly important because many interesting empirical questions or the zero lower bound (ZLB) of nominal interest rates lead directly to these type of situations. Furthermore, we can rely on a variety of well-tested numerical methods to solve the model and the associated continuous-time Hamilton-Jacobi-Bellman (HJB) equation. One key aspect of the HJB equation is that, thanks to the properties of stochastic calculus, it is deterministic even when we have underlying uncertainty. Thus, since we do not need to compute expectations (a burdensome step in discrete time Bellman equations), the solution of the model is faster and much simpler.

Motivated by these arguments, we show how to formulate and solve nonlinearly an otherwise standard NK model in continuous time. The basic structure of the economy is as follows. A representative household consumes, saves, and supplies labor. The final output is assembled by a final good producer, which uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent labor to manufacture their good. Also, these intermediate good producers face the constraint that they can only change prices following a Calvo's pricing rule. Finally, there is a government that fixes the one-period nominal interest rate through open market operations with public debt. In addition, the government taxes and consumes. We will have four shocks: one to preferences (which can be loosely interpreted as a shock to aggregate demand), one to technology (interpreted as a shock to aggregate supply), one to monetary policy, and one to fiscal policy. Then, we will show how the equilibrium system can be written in terms of 8 state variables. Our nonlinear and global numerical solution technique allows us to compute equilibrium dynamics and impulse response functions in the time space, the collocation method is based on Chebychev polynomials to compute the HJB equation and thus the solution in the policy function space.

We do not advocate the use of continuous time over discrete time in all cases and applications. Both approaches are sensible and the choice of one versus the other should depend on the application and the insights we get from it and not from any *a priori* positioning. This paper merely aims at expanding the set of tools available to researchers by showing how, in a real life example, we can handle rich models in macroeconomics using continuous time.

The rest of the paper is organized as follows. First, we present a simple NK model in continuous time and derive the HJB equation of the household. In Section 3, we define the equilibrium of the economy. Section 4 summarizes some analytical results. Section 5 analyzes the equilibrium dynamics and holds our main results on the effects of the zero lower bound on macro dynamics. Section 6 describes our numerical solution method in the policy function space, Section 7 holds some numerical results. We complete the paper with a description of the estimation process and with some final remarks. An appendix offers further details on some technical aspects of the paper.

2. Our Model

We describe now the environment that we use for our investigation. It is a rather straightforward NK model except for the continuous structure of time.

2.1. Households

There is a representative household in the economy that maximizes the following lifetime utility function, which is separable in consumption, c_t and hours worked, l_t:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \mathrm{d}t \tag{1}$$

where ρ is the subjective rate of time preference, ϑ is the inverse of Frisch labor supply elasticity, and d_t is a preference shock whose log follows an Ornstein-Uhlenbeck process:

$$\mathrm{d} \log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t} \tag{2}$$

where $B_{d,t}$ is a standard Brownian motion (also Wiener's process), or, by Itô's lemma:

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}.$$

Below, for this shock and the other two, we will use both the formulation in level and in logs depending on the context and easiness of notation.

The household can trade on Arrow securities (which we exclude to save on notation) and on a nominal government bonds b_t at a nominal interest rate of r_t (fixed coupon payments). Let n_t denote the number of shares and p_t^b the equilibrium price of bonds. Suppose the household earns a disposable income of $r_t b_t + p_t w_t l_t + p_t T_t + p_t \digamma_t$, where p_t is the price level (price of the consumption good), w_t is the real wage, T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy; the household's budget constraint is:

$$\mathrm{d}n_t = \frac{r_t b_t - p_t c_t + p_t w_t l_t + p_t T_t + p_t \digamma_t}{p_t^b}\mathrm{d}t. \tag{3}$$

Let bond prices follow:

$$\mathrm{d}p_t^b = \alpha_t p_t^b \mathrm{d}t \tag{4}$$

in which α_t denotes the endogenous rate of change, which is determined in general equilibrium (in equilibrium prices are function of the state variables, for example, by fixing α_t the bond supply has to accommodate so as to permit the bond's nominal interest rate being admissible). The household's financial wealth, $b_t = n_t p_t^b$, is then given by:

$$\mathrm{d}b_t = (r_t b_t - p_t c_t + p_t w_t l_t + p_t T_t + p_t \digamma_t)\mathrm{d}t + \alpha_t b_t \mathrm{d}t, \tag{5}$$

Let prices p_t follow the process:

$$\mathrm{d}p_t = \pi_t p_t \mathrm{d}t \tag{6}$$

such that the (realized) rate of inflation is locally non-stochastic. We can interpret $\mathrm{d}p_t/p_t$ as the realized inflation over the period $[t, t + \mathrm{d}t]$ and π_t as the inflation rate.[1]

Letting $a_t \equiv b_t/p_t$ denote real financial wealth and using Itô's formula, the household's real wealth evolves according to:

$$\mathrm{d}a_t = \frac{\mathrm{d}b_t}{p_t} - \frac{b_t}{p_t^2}\mathrm{d}p_t = \frac{r_t b_t - p_t c_t + p_t w_t l_t + p_t T_t + p_t \digamma_t + \alpha_t b_t}{p_t}\mathrm{d}t - \frac{b_t}{p_t^2}\pi_t p_t \mathrm{d}t$$

or:

$$\mathrm{d}a_t = ((r_t + \alpha_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\,\mathrm{d}t \tag{7}$$

2.2. The Final Good Producer

There is one final good is produced using intermediate goods with the following production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{8}$$

where ε is the elasticity of substitution.

[1] As it turns out below, we can just set $\alpha_t = 0$ if we require that $a_t = 0$ for all t. Our analysis, however, is not necessarily restricted to the case of no government liabilities. In case of government debt, $\alpha_t = \pi_t$ is required to keep government liabilities constant in real terms for the specified fiscal rule below.

Final good producers are perfectly competitive and maximize profits subject to the production function (8), taking as given all intermediate goods prices p_{it} and the final good price p_t. As a consequence the input demand functions associated with this problem are:

$$y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t \qquad \forall i,$$

and

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon}\mathrm{d}i\right)^{\frac{1}{1-\varepsilon}}. \tag{9}$$

2.3. Intermediate Good Producers

Each intermediate firm produces differentiated goods out of labor using:

$$y_{it} = A_t l_{it}$$

where l_{it} is the amount of the labor input rented by the firm and where A_t follows:

$$\mathrm{d}\log A_t = -\rho_a \log A_t \mathrm{d}t + \sigma_a \mathrm{d}B_{a,t}. \tag{10}$$

Therefore, the real marginal cost of the intermediate good producer is the same across firms:

$$mc_t = w_t/A_t.$$

The monopolistic firms engage in infrequent price setting á la Calvo. We assume that intermediate good producers reoptimize their prices p_{it} only at the time when a price-change signal is received. The probability (density) of receiving such a signal h periods from today is assumed to be independent of the last time the firm got the signal, and to be given by:

$$\delta e^{-\delta h}, \quad \delta > 0.$$

A number of firms δ will receive the price-change signal per unit of time. All other firms keep their old prices. Therefore, prices are set to maximize the expected discounted profits:

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-\delta(\tau-t)} \left(\frac{p_{it}}{p_\tau} y_{i\tau} - mc_\tau y_{i\tau}\right) \mathrm{d}\tau$$
$$\text{s.t. } y_{i\tau} = \left(\frac{p_{it}}{p_\tau}\right)^{-\varepsilon} y_\tau,$$

where λ_τ is the time t value of a unit of consumption in period τ to the household that value future prices from the perspective of the household (hence, the pricing kernel for the firm). Observe that $e^{-\delta(\tau-t)}$ denotes the probability of not having received a signal during $\tau - t$,

$$1 - \int_t^\tau \delta e^{-\delta(h-t)} dh = 1 - \left(-e^{-\delta(\tau-t)} + 1\right) = e^{-\delta(\tau-t)}. \tag{11}$$

After dropping constants, the first-order condition reads:

$$\mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)}(1-\varepsilon)\left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} p_{it} y_\tau \mathrm{d}\tau + \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} mc_\tau \varepsilon \left(\frac{p_t}{p_\tau}\right)^{-\varepsilon} p_t y_\tau \mathrm{d}\tau = 0.$$

We may write the first-order condition as:

$$p_{it} x_{1,t} = \frac{\varepsilon}{\varepsilon - 1} p_t x_{2,t} \quad \Rightarrow \quad \Pi_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{x_{2,t}}{x_{1,t}} \tag{12}$$

in which $\Pi_t^* \equiv p_{it}/p_t$ is the ratio between the optimal new price (common across all firms that can reset their prices) and the price of the final good and where we have defined the auxiliary variables:

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} \left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} y_\tau \mathrm{d}\tau,$$
$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} mc_\tau \left(\frac{p_t}{p_\tau}\right)^{-\varepsilon} y_\tau \mathrm{d}\tau,$$

Differentiating $x_{1,t}$ with respect to time gives:

$$\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t} &= e^{\delta t} p_t^{1-\varepsilon} \frac{1}{\mathrm{d}t} \mathrm{d}\mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta\tau} \left(\frac{1}{p_\tau}\right)^{1-\varepsilon} y_\tau \mathrm{d}\tau + \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta\tau} \left(\frac{1}{p_\tau}\right)^{1-\varepsilon} y_\tau \mathrm{d}\tau \frac{1}{\mathrm{d}t} \mathrm{d}\left(e^{\delta t} p_t^{1-\varepsilon}\right) \\
&= -\lambda_t y_t + \left(\delta e^{\delta t} p_t^{1-\varepsilon} + e^{\delta t}(1-\varepsilon) p_t^{1-\varepsilon} \frac{1}{\mathrm{d}t} \frac{\mathrm{d}p_t}{p_t}\right) \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta\tau} \left(\frac{1}{p_\tau}\right)^{1-\varepsilon} y_\tau \mathrm{d}\tau \\
&= -\lambda_t y_t + (\delta + (1-\varepsilon)\pi_t)\, \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-\delta(\tau-t)} \left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon} y_\tau \mathrm{d}\tau \\
&= -\lambda_t y_t + (\delta + (1-\varepsilon)\pi_t)\, x_{1,t}
\end{aligned}$$

or

$$\mathrm{d}x_{1,t} = \left((\delta + (1-\varepsilon)\pi_t)\, x_{1,t} - \lambda_t y_t\right) \mathrm{d}t \tag{13}$$

were we identify the actual rate of inflation π_t over the period $[t, t + \mathrm{d}t]$ with $\mathrm{d}p_t/p_t$. We can also renormalize $\lambda_t = e^{\rho t} m_t$ and get:

$$\mathrm{d}x_{1,t} = \left((\delta + (1-\varepsilon)\pi_t)\, x_{1,t} - e^{\rho t} m_t y_t\right) \mathrm{d}t$$

A similar procedure delivers:

$$\mathrm{d}x_{2,t} = \left((\delta - \varepsilon\pi_t)\, x_{2,t} - e^{\rho t} m_t mc_t y_t\right) \mathrm{d}t \tag{14}$$

Assuming that the price-change is stochastically independent across firms gives:

$$p_t^{1-\varepsilon} = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} p_{i\tau}^{1-\varepsilon} \, \mathrm{d}\tau,$$

making the price level p_t a predetermined variable at time t, its level being given by past price quotations (Calvo's insight). Differentiating with respect to time gives:

$$\begin{aligned} \mathrm{d}p_t^{1-\varepsilon} &= \left(\delta p_{it}^{1-\varepsilon} - \delta \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} p_{i\tau}^{1-\varepsilon} \mathrm{d}\tau \right) \mathrm{d}t \\ &= \delta \left(p_{it}^{1-\varepsilon} - p_t^{1-\varepsilon} \right) \mathrm{d}t \end{aligned}$$

and

$$\frac{1}{\mathrm{d}t} \mathrm{d}p_t^{1-\varepsilon} = (1-\varepsilon)\, p_t^{-\varepsilon} \frac{\mathrm{d}p_t}{\mathrm{d}t}.$$

Then

$$\mathrm{d}p_t = \frac{\delta}{1-\varepsilon} \left(p_{it}^{1-\varepsilon} p_t^{\varepsilon} - p_t \right) \mathrm{d}t \quad \Rightarrow \quad \pi_t = \frac{\delta}{1-\varepsilon} \left((\Pi_t^*)^{1-\varepsilon} - 1 \right). \tag{15}$$

Differentiating the previous expression, we obtain the inflation dynamics:

$$\begin{aligned} \frac{1}{\mathrm{d}t} \mathrm{d}\pi_t &= \delta \left(\Pi_t^* \right)^{-\varepsilon} \frac{1}{\mathrm{d}t} \mathrm{d}\Pi_t^* = \delta \left(\Pi_t^* \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} \frac{1}{\mathrm{d}t} \mathrm{d} \left(\frac{x_{2,t}}{x_{1,t}} \right) \\ &= \delta \left(\Pi_t^* \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} \frac{1}{x_{1,t}} \left(\frac{1}{\mathrm{d}t} \mathrm{d}x_{2,t} - \frac{x_{2,t}}{x_{1,t}} \frac{1}{\mathrm{d}t} \mathrm{d}x_{1,t} \right) \\ &= \delta \left(\Pi_t^* \right)^{1-\varepsilon} \frac{1}{x_{2,t}} \left(\frac{1}{\mathrm{d}t} \mathrm{d}x_{2,t} - \frac{x_{2,t}}{x_{1,t}} \frac{1}{\mathrm{d}t} \mathrm{d}x_{1,t} \right) \\ &= \delta \left(\Pi_t^* \right)^{1-\varepsilon} \left(\frac{1}{x_{2,t}} \frac{1}{\mathrm{d}t} \mathrm{d}x_{2,t} - \frac{1}{x_{1,t}} \frac{1}{\mathrm{d}t} \mathrm{d}x_{1,t} \right) \\ &= \delta \left(\Pi_t^* \right)^{1-\varepsilon} \left(\frac{\left((\delta - \varepsilon \pi_t)\, x_{2,t} - e^{\rho t} m_t mc_t y_t \right)}{x_{2,t}} - \frac{\left((\delta + (1-\varepsilon)\, \pi_t)\, x_{1,t} - e^{\rho t} m_t y_t \right)}{x_{1,t}} \right) \\ &= -\delta \left(\Pi_t^* \right)^{1-\varepsilon} \left(\pi_t + \left(\frac{mc_t}{x_{2,t}} - \frac{1}{x_{1,t}} \right) e^{\rho t} m_t y_t \right). \end{aligned} \tag{16}$$

2.4. The Government Problem

The government sets the nominal interest rate r_t through open market operations according to the Taylor rule (similar to Sims 2004, p.291):

$$\mathrm{d}r_t = (\theta_0 + \theta_1 \pi_t - \theta_2 r_t)\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}, \tag{17}$$

The monetary authority buys or sells government bonds such as the nominal interest rate follows (17) and the bond market clears (government bond supply now is endogenous). This rule reflects both a response to inflation through the parameter θ_1 and a desire to smooth interest rates over time through θ_2. The constant $\theta_0 \equiv \theta_2 r_{ss} - \theta_1 \pi_{ss}$ summarizes the attitude of the monetary authority towards either the average nominal interest rate or the target of inflation (one target is isomorphic to the other, but both cannot be selected simultaneously since we are dealing with a general equilibrium model). Moreover, the term σ_m specifies the variance of shocks to monetary policy.

The coupon payments of the government perpetuities $T_t^b = -r_t a_t$ are financed through lump-sum taxes. Suppose transfers finance a given stream of government consumption expressed in terms of its constant share of output, $s_g s_{g,t}$, with a mean s_g and a stochastic component $s_{g,t}$ that follows another Ornstein-Uhlenbeck process[2]:

$$\mathrm{d}\log s_{g,t} = -\rho_g \log s_{g,t}\mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}, \tag{18}$$

such that

$$g_t - T_t^b = s_g s_{g,t} y_t - T_t^b \equiv -T_t.$$

2.5. Aggregation

First, we derive an expression for aggregate demand:

$$y_t = c_t + g_t.$$

In other words, there is no possibility to transfer the output good intertemporally. With this value, the demand for each intermediate good producer is

$$y_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \qquad \forall i. \tag{19}$$

Using the production function we may write:

$$A_t l_{it} = (c_t + g_t)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon}.$$

We can integrate on both sides:

$$A_t \int_0^1 l_{it}\mathrm{d}i = (c_t + g_t)\int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i$$

and get an expression:

$$c_t + g_t = y_t = \frac{A_t}{v_t} l_t$$

where

$$v_t = \int_0^1 \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i \tag{20}$$

[2]While we could have $s_g s_{g,t} > 1$, our calibration of s_g and σ_g is such that this event will happen with a negligibly small probability. Alternatively we could specify a stochastic process with support (0,1).

is the aggregate loss of efficiency induced by price dispersion of the intermediate goods. Similar to the price level, v_t is a predetermined variable (Calvo's insight):

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} \mathrm{d}\tau.$$

Differentiating with respect to time gives:

$$\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}v_t &= \delta\left(\Pi_t^*\right)^{-\varepsilon} + \int_{-\infty}^{t} \delta \frac{1}{\mathrm{d}t}\mathrm{d}e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} \mathrm{d}\tau \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon} - \delta \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon} \mathrm{d}\tau + \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} p_{i\tau}^{-\varepsilon} \frac{1}{\mathrm{d}t}\mathrm{d}p_t^{\varepsilon}\mathrm{d}\tau \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon} - \delta v_t + \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} p_{i\tau}^{-\varepsilon} \varepsilon p_t^{\varepsilon-1} \frac{1}{\mathrm{d}t}\mathrm{d}p_t \mathrm{d}\tau \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t. \qquad (21)
\end{aligned}$$

For aggregate profits, we use the demand of intermediate producers in (19):

$$\begin{aligned}
F_t &= \int_0^1 \left(\frac{p_{it}}{p_t} - mc_t\right) y_{it}\mathrm{d}i \\
&= y_t \int_0^1 \left(\frac{p_{it}}{p_t} - mc_t\right)\left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i \\
&= \left(\int_0^1 \left(\frac{p_{it}}{p_t}\right)^{1-\varepsilon} \mathrm{d}i - mc_t v_t\right) y_t \\
&= (1 - mc_t v_t) y_t. \qquad (22)
\end{aligned}$$

2.6. The HJB Equation First-Order Conditions

Given our description of the problem, we define the household's value function as:

$$V(\mathbb{Z}_t; \mathbb{Y}_t) \equiv \max_{\{(c_t, l_t)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} d_t \left\{\log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}\right\} dt$$

s.t.

$$
\begin{aligned}
\mathrm{d}a_t &= ((r_t + \alpha_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\,\mathrm{d}t \ (7) \\
\mathrm{d}r_t &= (\theta_0 + \theta_1 \pi_t - \theta_2 r_t)\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t} \ (17) \\
\mathrm{d}v_t &= \left(\delta\left(\Pi_t^*\right)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right)\mathrm{d}t \ (21) \\
\mathrm{d}x_{1,t} &= \left((\delta - (\varepsilon - 1)\pi_t)x_{1,t} - e^{\rho t} m_t y_t\right)\mathrm{d}t \ (13) \\
\mathrm{d}x_{2,t} &= \left((\delta - \varepsilon\pi_t)\,x_{2,t} - e^{\rho t} m_t y_t mc_t\right)\mathrm{d}t \ (14) \\
\mathrm{d}\log d_t &= -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t} \ (2) \\
\mathrm{d}\log A_t &= -\rho_a \log A_t \mathrm{d}t + \sigma_a \mathrm{d}B_{a,t} \ (10) \\
\mathrm{d}\log s_{g,t} &= -\rho_g \log s_{g,t} \mathrm{d}t + \sigma_g \mathrm{d}B_{g,t} \ (18),
\end{aligned}
$$

in which we define the vector of relevant state variables $\mathbb{Z}_t \equiv (a_t, r_t, v_t, x_{1,t}, x_{2,t}, d_t, A_t, s_{g,t})$ and $\mathbb{Y}_t \equiv (y_t, mc_t, w_t, \pi_t, \Pi_t^*, m_t, T_t, \digamma_t) = \mathbb{Y}(\mathbb{Z}_t)$ to be determined in equilibrium, so far taken as parametric by the household. By choosing the control $(c_t, l_t) \in \mathbb{R}_+^2$ at time t, the HJB equation reads:

$$
\begin{aligned}
\rho V(\mathbb{Z}_t; \mathbb{Y}_t) \quad = \quad & \max_{(c_t, l_t)} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \\
& + ((r_t + \alpha_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\, V_a \\
& + (\theta_0 + \theta_1 \pi_t - \theta_2 r_t) V_r + \tfrac{1}{2}\sigma_m^2 V_{rr} \\
& + \left(\delta\left(\Pi_t^*\right)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right) V_v \\
& + \left((\delta - (\varepsilon - 1)\pi_t)x_{1,t} - e^{\rho t} m_t y_t\right) V_{x_1} \\
& + \left((\delta - \varepsilon\pi_t)\,x_{2,t} - e^{\rho t} m_t y_t mc_t\right) V_{x_2} \\
& - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_d + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dd} \\
& - (\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2) A_t V_A + \tfrac{1}{2}\sigma_a^2 A_t^2 V_{AA} \\
& - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_s + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ss}. \qquad (23)
\end{aligned}
$$

A neat result about the formulation of our problem in continuous time is that the HJB equation is, in effect, a deterministic differential equation.

The first-order conditions with respect to c_t and l_t for any interior solution are:

$$
\begin{aligned}
\frac{d_t}{c_t} &= V_a \qquad (24) \\
d_t \psi l_t^{\vartheta} &= V_a w_t \qquad (25)
\end{aligned}
$$

or, eliminating the costate variable (for $\psi \neq 0$):

$$
\psi l_t^{\vartheta} c_t = w_t
$$

which is the standard static optimality condition between labor and consumption.

The first-order conditions (24) and (25) make the optimal controls functions of the state variables, $c_t = c(\mathbb{Z}_t; \mathbb{Y}_t)$, $l_t = l(\mathbb{Z}_t; \mathbb{Y}_t)$. Thus, the concentrated HJB equation reads:

$$\begin{aligned}
\rho V(\mathbb{Z}_t; \mathbb{Y}_t) \;\; = \;\; & d_t \log c(\mathbb{Z}_t; \mathbb{Y}_t) - d_t \psi \frac{l(\mathbb{Z}_t; \mathbb{Y}_t)^{1+\vartheta}}{1+\vartheta} \\
& + \left((r_t + \alpha_t - \pi_t) a_t - c(\mathbb{Z}_t; \mathbb{Y}_t) + w_t l(\mathbb{Z}_t; \mathbb{Y}_t) + T_t + F_t \right) V_a \\
& + (\theta_0 + \theta_1 \pi_t - \theta_2 r_t) V_r + \tfrac{1}{2} \sigma_m^2 V_{rr} \\
& + \left(\delta \left(\Pi_t^* \right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t \right) V_v \\
& + \left((\delta - (\varepsilon - 1) \pi_t) x_{1,t} - e^{\rho t} m_t y_t \right) V_{x_1} \\
& + \left((\delta - \varepsilon \pi_t) \, x_{2,t} - e^{\rho t} m_t y_t m c_t \right) V_{x_2} \\
& - (\rho_d \log d_t - \tfrac{1}{2} \sigma_d^2) d_t V_d + \tfrac{1}{2} \sigma_d^2 d_t^2 V_{dd} \\
& - (\rho_a \log A_t - \tfrac{1}{2} \sigma_a^2) A_t V_A + \tfrac{1}{2} \sigma_a^2 A_t^2 V_{AA} \\
& - (\rho_g \log s_{g,t} - \tfrac{1}{2} \sigma_g^2) s_{g,t} V_s + \tfrac{1}{2} \sigma_g^2 s_{g,t}^2 V_{ss}.
\end{aligned} \tag{26}$$

Using the envelope theorem, we obtain the costate variable V_a as:

$$\begin{aligned}
\rho V_a \;\; = \;\; & (r_t + \alpha_t - \pi_t) V_a + \left((r_t + \alpha_t - \pi_t) a_t - c_t + w_t l_t + T_t + F_t \right) V_{aa} \\
& + (\theta_0 + \theta_1 \pi_t - \theta_2 r_t) V_{ra} + \tfrac{1}{2} \sigma_m^2 V_{rra} \\
& + \left(\delta \left(\Pi_t^* \right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t \right) V_{va} \\
& + \left((\delta - (\varepsilon - 1) \pi_t) x_{1,t} - e^{\rho t} m_t y_t \right) V_{x_1 a} \\
& + \left((\delta - \varepsilon \pi_t) \, x_{2,t} - e^{\rho t} m_t y_t m c_t \right) V_{x_2 a} \\
& - (\rho_d \log d_t - \tfrac{1}{2} \sigma_d^2) d_t V_{da} + \tfrac{1}{2} \sigma_d^2 d_t^2 V_{dda} \\
& - (\rho_a \log A_t - \tfrac{1}{2} \sigma_a^2) A_t V_{Aa} + \tfrac{1}{2} \sigma_a^2 A_t^2 V_{AAa} \\
& - (\rho_g \log s_{g,t} - \tfrac{1}{2} \sigma_g^2) s_{g,t} V_{sa} + \tfrac{1}{2} \sigma_g^2 s_{g,t}^2 V_{ssa}.
\end{aligned} \tag{27}$$

An alternative formulation in terms of differentials is:

$$\begin{aligned}
(\rho - r_t - \alpha_t + \pi_t) \, V_a \mathrm{d}t \;\; = \;\; & V_{aa} \mathrm{d}a_t + (\mathrm{d}r_t - \sigma_m \mathrm{d}B_{m,t}) V_{ra} + \tfrac{1}{2} \sigma_m^2 V_{rra} + V_{va} \mathrm{d}v_t \\
& + V_{x_1 a} \mathrm{d}x_{1,t} + V_{x_2 a} \mathrm{d}x_{2,t} + (\mathrm{d}d_t - \sigma_d d_t \mathrm{d}B_{d,t}) \, V_{da} + \tfrac{1}{2} \sigma_d^2 d_t^2 V_{dda} \mathrm{d}t \\
& + (\mathrm{d}A_t - \sigma_a A_t \mathrm{d}B_{a,t}) \, V_{Aa} + \tfrac{1}{2} \sigma_a^2 A_t^2 V_{AAa} \mathrm{d}t + (\mathrm{d}s_{g,t} - \sigma_g s_{g,t} \mathrm{d}B_{g,t}) \, V_{sa} + \tfrac{1}{2} \sigma_g^2 s_{g,t}^2 V_{ssa} \mathrm{d}t
\end{aligned}$$

or

$$\begin{aligned}
& (\rho - r_t - \alpha_t + \pi_t) \, V_a \mathrm{d}t + \sigma_d d_t V_{da} \mathrm{d}B_{d,t} + \sigma_a A_t V_{Aa} \mathrm{d}B_{a,t} + \sigma_g s_{g,t} V_{sa} \mathrm{d}B_{g,t} + \sigma_m r_t V_{ra} \mathrm{d}B_{m,t} \\
= \; & V_{aa} \mathrm{d}a_t + V_{ra} \mathrm{d}r_t + \tfrac{1}{2} \sigma_m^2 r_t^2 V_{rra} + V_{va} \mathrm{d}v_t + V_{x_1 a} \mathrm{d}x_{1,t} + V_{x_2 a} \mathrm{d}x_{2,t} \\
& + V_{da} \mathrm{d}d_t + \tfrac{1}{2} \sigma_d^2 d_t^2 V_{da} \mathrm{d}t + V_{Aa} \mathrm{d}A_t + \tfrac{1}{2} \sigma_a^2 A_t^2 V_{Aa} \mathrm{d}t + V_{sa} \mathrm{d}s_{g,t} + \tfrac{1}{2} \sigma_g^2 s_{g,t}^2 V_{sa} \mathrm{d}t.
\end{aligned}$$

Observe that the costate variable in general evolves according to:

$$
\begin{aligned}
\mathrm{d}V_a &= V_{aa}\mathrm{d}a_t + V_{ra}\mathrm{d}r_t + \tfrac{1}{2}\sigma_m^2 V_{rra}\mathrm{d}t + V_{va}\mathrm{d}v_t + V_{x_1a}\mathrm{d}x_{1,t} + V_{x_2a}\mathrm{d}x_{2,t} \\
&\quad + V_{da}\mathrm{d}d_t + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dda}\mathrm{d}t + V_{Aa}\mathrm{d}A_t + \tfrac{1}{2}\sigma_a^2 A_t^2 V_{AAa}\mathrm{d}t + V_{sa}\mathrm{d}s_{gt} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ssa}\mathrm{d}t \\
&= (\rho - r_t - \alpha_t + \pi_t)V_a\mathrm{d}t \\
&\quad + \sigma_d d_t V_{da}\mathrm{d}B_{d,t} + \sigma_a A_t V_{Aa}\mathrm{d}B_{a,t} + \sigma_g s_{g,t} V_{sa}\mathrm{d}B_{g,t} + \sigma_m V_{ra}\mathrm{d}B_{m,t}, \qquad (28)
\end{aligned}
$$

Note that (28) determines the stochastic discount factor (SDF) consistent with equilibrium dynamics of macro aggregates, which can be used to price any asset in the economy:

$$
\begin{aligned}
\mathrm{d}\ln V_a &= \frac{1}{V_a}\mathrm{d}V_a - \tfrac{1}{2}\sigma_d^2 d_t^2 \frac{V_{da}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_a^2 A_t^2 \frac{V_{Aa}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 \frac{V_{sa}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_m^2 \frac{V_{ra}^2}{V_a^2}\mathrm{d}t \\
&= (\rho - r_t - \alpha_t + \pi_t)\mathrm{d}t + \sigma_d d_t \frac{V_{da}}{V_a}\mathrm{d}B_{d,t} + \sigma_a A_t \frac{V_{Aa}}{V_a}\mathrm{d}B_{a,t} + \sigma_g s_{g,t} \frac{V_{sa}}{V_a}\mathrm{d}B_{g,t} + \sigma_m \frac{V_{ra}}{V_a}\mathrm{d}B_{m,t} \\
&\quad - \tfrac{1}{2}\sigma_d^2 d_t^2 \frac{V_{da}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_a^2 A_t^2 \frac{V_{Aa}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 \frac{V_{sa}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_m^2 \frac{V_{ra}^2}{V_a^2}\mathrm{d}t.
\end{aligned}
$$

For $s > t$, we may write:

$$
e^{-\rho(s-t)}\frac{V_a(\mathbb{Z}_s;\mathbb{Y}_t)}{V_a(\mathbb{Z}_t;\mathbb{Y}_t)} =
$$

$$
\exp\left(
\begin{array}{c}
-\int_t^s (r_u + \alpha_u - \pi_u)\mathrm{d}u - \frac{1}{2}\int_t^s \frac{V_{da}^2}{V_a^2}\sigma_d^2 d_u^2 \mathrm{d}u - \frac{1}{2}\int_t^s \frac{V_{Aa}^2}{V_a^2}\sigma_a^2 A_u^2 \mathrm{d}u \\
-\frac{1}{2}\int_t^s \frac{V_{sa}^2}{V_a^2}\sigma_g^2 s_{g,u}^2 \mathrm{d}u - \frac{1}{2}\int_t^s \frac{V_{ra}^2}{V_a^2}\sigma_m^2 \mathrm{d}u \\
+\int_t^s \frac{V_{da}}{V_a}\sigma_d d_u \mathrm{d}B_{d,u} + \int_t^s \frac{V_{Aa}}{V_a}\sigma_a A_u \mathrm{d}B_{a,u} + \int_t^s \frac{V_{sa}}{V_a}\sigma_g s_{g,u}\mathrm{d}B_{g,u} + \int_t^s \frac{V_{ra}}{V_a}\sigma_m \mathrm{d}B_{m,u}
\end{array}
\right).
$$

Hence, the implied SDF is (see Hansen and Scheinkmann, 2009):

$$
\frac{m_s}{m_t} = e^{-\rho(s-t)}\frac{V_a(\mathbb{Z}_s;\mathbb{Y}_s)}{V_a(\mathbb{Z}_t;\mathbb{Y}_t)},
$$

and we can pin down $m_t = e^{-\rho t}V_a(\mathbb{Z}_t;\mathbb{Y}_t)$.

Using the first-order condition (24) and (28), we obtain the implicit Euler equation:

$$
\mathrm{d}\left(\frac{d_t}{c_t}\right) = (\rho - r_t - \alpha_t + \pi_t)\left(\frac{d_t}{c_t}\right)\mathrm{d}t
$$
$$
-\sigma_d d_t \frac{d_t}{c_t^2}c_d\mathrm{d}B_{d,t} - \sigma_a A_t \frac{d_t}{c_t^2}c_A\mathrm{d}B_{a,t} - \sigma_g s_{g,t}\frac{d_t}{c_t^2}c_s\mathrm{d}B_{g,t} - \sigma_m \frac{d_t}{c_t^2}c_r\mathrm{d}B_{m,t},
$$

in which $V_{ad} = -\left(d_t/c_t^2\right)c_d$, $V_{Aa} = -\left(d_t/c_t^2\right)c_A$, $V_{sa} = -\left(d_t/c_t^2\right)c_s$, and $V_{ra} = -\left(d_t/c_t^2\right)c_r$ are expressed in terms of derivatives with respect to the optimal consumption function. This equation has a simple interpretation: the change in the marginal utility of consumption depends on the rate of time preference minus the effective real interest rate and four additional terms that control for the innovations to the four shocks to the economy.

Hence, by applying Itô's formula we obtain the Euler equation:

$$
\begin{aligned}
\mathrm{d}\left(\frac{c_t}{d_t}\right) = -\left(\frac{d_t}{c_t}\right)^{-1}(\rho - r_t - \alpha_t + \pi_t)\mathrm{d}t + \sigma_d^2 \frac{d_t}{c_t} c_d^2 \mathrm{d}t + \sigma_a^2 A_t^2 \frac{d_t^{-1}}{c_t} c_A^2 \mathrm{d}t + \sigma_g^2 s_{g,t}^2 \frac{d_t^{-1}}{c_t} c_s^2 \mathrm{d}t \\
+\sigma_m^2 \frac{d_t^{-1}}{c_t} c_r^2 \mathrm{d}t + \sigma_d c_d \mathrm{d}B_{d,t} + \sigma_a A_t d_t^{-1} c_A \mathrm{d}B_{a,t} + \sigma_g s_{g,t} d_t^{-1} c_s \mathrm{d}B_{g,t} + \sigma_m d_t^{-1} c_r \mathrm{d}B_{m,t}
\end{aligned}
$$

or

$$
\begin{aligned}
\mathrm{d}c_t \quad = \quad & -(\rho - r_t - \alpha_t + \pi_t)c_t \mathrm{d}t + \sigma_d^2 \frac{d_t^2}{c_t} c_d^2 \mathrm{d}t + \sigma_a^2 \frac{A_t^2}{c_t} c_A^2 \mathrm{d}t + \sigma_g^2 \frac{s_{g,t}^2}{c_t} c_s^2 \mathrm{d}t + \sigma_m^2 \frac{1}{c_t} c_r^2 \mathrm{d}t \\
& +\sigma_d c_d d_t \mathrm{d}B_{d,t} + \sigma_a A_t c_A \mathrm{d}B_{a,t} + \sigma_g s_{g,t} c_s \mathrm{d}B_{g,t} + \sigma_m c_r \mathrm{d}B_{m,t} \\
& -c_t \rho_d \log d_t \mathrm{d}t + c_t \sigma_d \mathrm{d}B_{d,t} + \tfrac{1}{2} c_t \sigma_d^2 \mathrm{d}t + \sigma_d^2 d_t c_d \mathrm{d}t, \qquad (29)
\end{aligned}
$$

where $c = c(\mathbb{Z}_t; \mathbb{Y}_t)$ denotes the household's consumption function for a given $\mathbb{Y}_t$.

3. Equilibrium

The general equilibrium is given by the sequence $\{c_t, l_t, a_t, mc_t, x_{1,t}, x_{2,t}, \digamma_t, w_t, r_t, g_t, T_t, \pi_t, \Pi_t^*, v_t, y_t, d_t, A_t, s_{g,t}\}_{t=0}^{\infty}$ determined by the following equations:

- Euler equation, the first-order conditions of the household, and budget constraint:

Equation 1

$$
\begin{aligned}
\mathrm{d}c_t = -(\rho - r_t - \alpha_t + \pi_t)c_t \mathrm{d}t + \sigma_d^2 \frac{d_t^2}{c_t} c_d^2 \mathrm{d}t + \sigma_a^2 \frac{A_t^2}{c_t} c_A^2 \mathrm{d}t + \sigma_g^2 \frac{s_{g,t}^2}{c_t} c_s^2 \mathrm{d}t + \sigma_m^2 \frac{1}{c_t} c_r^2 \mathrm{d}t \\
-c_t \rho_d \log d_t \mathrm{d}t + \tfrac{1}{2} c_t \sigma_d^2 \mathrm{d}t + \sigma_d^2 d_t c_d \mathrm{d}t \\
+(\sigma_d c_d d_t + c_t \sigma_d)\mathrm{d}B_{d,t} + \sigma_a A_t c_A \mathrm{d}B_{a,t} + \sigma_g s_{g,t} c_s \mathrm{d}B_{g,t} + \sigma_m c_r \mathrm{d}B_{m,t}
\end{aligned}
$$

Equation 2

$$\psi l_t^{\vartheta} c_t = w_t$$

Equation 3

$$d_t / c_t = V_a$$

(redundant)

$$\mathrm{d}a_t = ((r_t + \alpha_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\, \mathrm{d}t$$

- Profit maximization is given by:

Equation 4

$$\Pi_t^* = \frac{\varepsilon}{\varepsilon - 1}\frac{x_{2,t}}{x_{1,t}}$$

Equation 5

$$\mathrm{d}x_{1,t} = ((\delta - (\varepsilon - 1)\pi_t)x_{1,t} - V_a y_t)\,\mathrm{d}t$$

Equation 6

$$\mathrm{d}x_{2,t} = ((\delta - \varepsilon\pi_t)\,x_{2,t} - V_a y_t mc_t)\,\mathrm{d}t$$

Equation 7

$$\digamma_t = (1 - mc_t v_t)y_t$$

Equation 8

$$w_t = A_t mc_t$$

- Government policy:

Equation 9

$$\mathrm{d}r_t = (\theta_0 + \theta_1\pi_t - \theta_2 r_t)\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}$$

Equation 10

$$g_t = s_g s_{g,t} y_t$$

(redundant)

$$T_t = -r_t a_t - s_g s_{g,t} y_t$$

- Inflation evolution and price dispersion:

Equation 11

$$\pi_t = \frac{\delta}{1 - \varepsilon}\left((\Pi_t^*)^{1-\varepsilon} - 1\right)$$

Equation 12

$$\mathrm{d}v_t = \left(\delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right)\mathrm{d}t$$

- Market clearing on goods and labor markets:

Equation 13

$$y_t = c_t + g_t \quad \text{(expenditure)}$$

Equation 14

$$y_t = \frac{A_t}{v_t} l_t \quad \text{(production)}$$

(redundant)

$$y_t = w_t l_t + \digamma_t \quad \text{(income)}$$

- Stochastic processes follow:

Equation 15

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}$$

Equation 16

$$\mathrm{d}A_t = -\left(\rho_a \log A_t - \tfrac{1}{2}\sigma_a^2\right) A_t \mathrm{d}t + \sigma_a A_t \mathrm{d}B_{a,t}$$

Equation 17

$$\mathrm{d}s_{g,t} = -\left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right) s_{g,t} dt + \sigma_g s_{g,t} \mathrm{d}B_{g,t}$$

Further, using the household's budget constraint, we get in equilibrium:

$$\begin{aligned} \mathrm{d}a_t &= ((r_t + \alpha_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\mathrm{d}t \\ &= ((\alpha_t - \pi_t)a_t - c_t - g_t + y_t)\mathrm{d}t \\ &= (\alpha_t - \pi_t)a_t \mathrm{d}t, \end{aligned}$$

where for $\mathrm{d}a_t = 0$ either $\alpha_t = \pi_t$ and/or $a_t = 0$ for all t. It illustrates that our fiscal rule requires α_t to offset the inflation rate (in what follows we use $a_t = 0$ and $\alpha_t = 0$).

4. Analytical results

Since we are interested in the general equilibrium dynamics for the case where $a_t = 0$, we use a short-cut approach and directly solve (26) with general equilibrium values for $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$. We compute $V(\mathbb{Z}_t; \mathbb{Y}_t)$ for $a_t = 0$ at values for $\mathbb{Y}(\mathbb{Z}_t)$ which clear all markets.

Steady-state. Without shocks the economy moves towards its steady state. Observe that non-stochastic steady-state values for the stochastic processes are $A_{ss} = d_{ss} = s_{g,ss} = 1$.

- Euler equation, the first-order conditions of the household, and budget constraint:

Equation 1

$$\pi_{ss} = r_{ss} - \rho$$

Equation 2

$$\psi l_{ss}^{\vartheta} c_{ss} = w_{ss}$$

Equation 3

$$c_{ss}^{-1} = V_a$$

- Profit maximization is given by:

Equation 4

$$\Pi^*_{ss} = \frac{\varepsilon}{\varepsilon - 1} \frac{x_{2,ss}}{x_{1,ss}}$$

Equation 5

$$0 = (\delta - (\varepsilon - 1)\pi_{ss})x_{1,ss} - V_a y_{ss}$$

Equation 6

$$0 = (\delta - \varepsilon \pi_t)\, x_{2,ss} - V_a y_{ss} mc_{ss}$$

Equation 7

$$F_{ss} = (1 - mc_{ss} v_{ss}) y_{ss}$$

Equation 8

$$w_{ss} = A_{ss} mc_{ss}$$

- Government policy:

Equation 9

(This equation is an identity in the steady state.)

Equation 10

$$g_{ss} = s_g y_{ss}$$

- Inflation evolution and price dispersion:

Equation 11

$$\pi_{ss} = \frac{\delta}{1 - \varepsilon} \left((\Pi^*_{ss})^{1-\varepsilon} - 1 \right)$$

Equation 12

$$0 = \delta \left(\Pi^*_{ss}\right)^{-\varepsilon} + (\varepsilon \pi_{ss} - \delta) v_{ss}$$

- Market clearing on goods and labor markets (one condition is redundant):

Equation 13

$$y_{ss} = c_{ss} + g_{ss} \quad \text{(expenditure)}$$

Equation 14

$$y_{ss} = \frac{A_{ss}}{v_{ss}} l_{ss} \quad \text{(production)}$$

(redundant)

$$y_{ss} = w_{ss} l_{ss} + F_{ss} \quad \text{(income)}$$

- Stochastic processes follow:

Equation 15

$$d_{ss} = 1$$

Equation 16

$$A_{ss} = 1$$

Equation 17

$$s_{g,ss} = 1$$

Given the level of steady-state inflation around which the model often is log-linearized we obtain the following steady-state values. Using **Equation 1**, we obtain

$$r_{ss} = \pi_{ss} + \rho$$

Using **Equation 11**, we obtain the steady-state value Π^*_{ss}

$$\Pi^*_{ss} = (1 - (\varepsilon - 1)(\pi_{ss}/\delta))^{\frac{1}{1-\varepsilon}}$$

Using **Equations 5** and **6** we can solve for the steady-state value of the marginal cost:

$$mc_{ss} = (\delta - \varepsilon\pi_{ss})\,(1 - s_g)x_{2,ss} \quad \text{or} \quad mc_{ss} = \frac{\delta - \varepsilon\pi_{ss}}{\delta - (\varepsilon - 1)\pi_{ss}} \frac{x_{2,ss}}{x_{1,ss}}$$

which by inserting **Equation 4** gives:

$$mc_{ss} = \frac{\delta - \varepsilon\pi_{ss}}{\delta - (\varepsilon - 1)\pi_{ss}} \frac{\varepsilon - 1}{\varepsilon} \Pi^*_{ss}$$

Hence, we obtain

$$x_{1,ss} = 1/\left((1-s_g)(\delta-(\varepsilon-1)\pi_{ss})\right)$$

Using **Equation 8**, we obtain

$$w_{ss} = mc_{ss}$$

Using **Equation 12** gives the steady-state value of price dispersion

$$v_{ss} = \frac{\delta\left(\Pi^*_{ss}\right)^{-\varepsilon}}{\delta-\varepsilon\pi_{ss}}$$

Using **Equation 14**, we obtain

$$y_{ss} = l_{ss}/v_{ss}$$

Using **Equation 13** and **Equation 10** yields

$$y_{ss} = c_{ss}/(1-s_g)$$

Combining the last two equations gives

$$l_{ss}/v_{ss} = c_{ss}/(1-s_g)$$

Using **Equation 2** we get

$$\psi l^{\vartheta}_{ss} c_{ss} = w_{ss}$$

hence we can collect terms to obtain

$$l_{ss} = \left(\frac{w_{ss}v_{ss}}{\psi(1-s_g)}\right)^{\frac{1}{1+\vartheta}}$$

Using **Equation 7** and **Equation 14** we get

$$F_{ss} = (1-mc_{ss}v_{ss})l_{ss}/v_{ss}$$

Note that the TVC requires that $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_0 V(\mathbb{Z}^*_t) = 0$, in which $\mathbb{Z}^*_t$ denotes the optimal state variables in line with general equilibrium conditions.

5. Equilibrium dynamics and impulse response analysis

In this section, we obtain the reduced-form equilibrium dynamics. First, we consider the non-linear system of stochastic differential equations which can be used to get impulse response functions. Second, for comparison we also consider the linear approximation of equilibrium dynamics around the non-stochastic steady state.

Equilibrium dynamics. It is instructive illustrate the mechanics of the new Keynesian model. We start with the combined first-order condition and market clearing:

$$\begin{aligned} w_t = \psi l_t^{\vartheta} c_t \;\Leftrightarrow\; v_t w_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t}) A_t \\ \Leftrightarrow\; l_t^{1+\vartheta} = \frac{v_t w_t}{(1 - s_g s_{g,t}) A_t \psi} \end{aligned}$$

or by (25)

$$c_t = ((1 - s_g s_{g,t})/v_t)^{\frac{\vartheta}{1+\vartheta}} A_t (mc_t/\psi)^{\frac{1}{1+\vartheta}}. \tag{30}$$

or

$$mc_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})/v_t$$

Our key result is that any (partial) equilibrium, i.e., any equilibrium for a given level of marginal cost, mc_t, the policy functions are available analytically. In the new Keynesian model, however, the firm takes into account expected future marginal cost *and* current marginal cost whenever it has an opportunity to adjust its price. As we show below, the equilibrium value for marginal costs is an unknown function of the states, $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$.

First, we insert the equilibrium SDF into the evolution of expectations. As it turns out, in equilibrium the discounted expected future profits, $x_{1,t}$ and costs $x_{2,t}$ are independent from the control variables. In equilibrium where markets clear the variables follow:

$$\begin{aligned} \mathrm{d}x_{1,t} &= \left((\Pi_t^*)^{1-\varepsilon}\delta x_{1,t} - e^{\rho t} m_t y_t\right)\mathrm{d}t \\ &= \left(((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1))^{1-\varepsilon}\,\delta x_{1,t} - e^{\rho t} m_t y_t\right)\mathrm{d}t \\ &= \left(((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1))^{1-\varepsilon}\,\delta x_{1,t} - d_t/(1 - s_g s_{g,t})\right)\mathrm{d}t \end{aligned}$$

and similarly:

$$\begin{aligned} \mathrm{d}x_{2,t} &= \left(\left(\varepsilon\,(\Pi_t^*)^{1-\varepsilon} - 1\right)\delta x_{2,t}/(\varepsilon-1) - e^{\rho t} m_t y_t mc_t\right)\mathrm{d}t \\ &= \left(\left(\varepsilon\,((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1))^{1-\varepsilon} - 1\right)\delta x_{2,t}/(\varepsilon-1) - e^{\rho t} m_t y_t mc_t\right)\mathrm{d}t \\ &= \left(((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1))^{1-\varepsilon}\,\delta x_{2,t}\varepsilon/(\varepsilon-1) - \delta x_{2,t}/(\varepsilon-1) - mc_t d_t/(1 - s_g s_{g,t})\right)\mathrm{d}t \end{aligned}$$

We are left with 5 non-linear differential equation, i.e., for the auxiliary variables $x_{1,t}$, $x_{2,t}$, price dispersion v_t, the Taylor rule r_t, and the Euler equation, which determines marginal costs, mc_t together with the 3 exogenous shock processes for $s_{g,t}, d_t, A_t$.

To summarize, denoting the shadow price $V_a = \lambda_t$ the equilibrium dynamics are:

Equation 1

$$\mathrm{d}\lambda_t = (\rho - r_t + \pi_t)\lambda_t \mathrm{d}t + \sigma_d d_t \lambda_d \mathrm{d}B_{d,t} + \sigma_a A_t \lambda_A \mathrm{d}B_{a,t} + \sigma_g s_{g,t} \lambda_s \mathrm{d}B_{g,t} + \sigma_m r_t \lambda_r \mathrm{d}B_{m,t}$$

Equation 5

$$\mathrm{d}x_{1,t} = ((\delta - (\varepsilon - 1)\pi_t)x_{1,t} - d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t$$

Equation 6

$$\mathrm{d}x_{2,t} = ((\delta - \varepsilon\pi_t)\, x_{2,t} - mc_t d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t$$

Equation 9

$$\mathrm{d}r_t = (\theta_0 + \theta_1 \pi_t - \theta_2 r_t)\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}$$

Equation 12

$$\mathrm{d}v_t = (\delta(1 + \pi_t(1 - \varepsilon)/\delta)^{\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \delta)v_t)\mathrm{d}t$$

in which $(1 + \pi_t(1 - \varepsilon)/\delta)^{\frac{1}{1-\varepsilon}} = \varepsilon/(\varepsilon - 1)(x_{2,t}/x_{1,t})$ determines the inflation rate and

$$\begin{aligned} \lambda_t &= ((1 - s_g s_{g,t})/v_t)^{-\frac{\vartheta}{1+\vartheta}} (mc_t/\psi)^{-\frac{1}{1+\vartheta}} d_t/A_t, \\ \Leftrightarrow\ mc_t &= \psi(\lambda_t(A_t/d_t))^{-(1+\vartheta)}(v_t/(1 - s_g s_{g,t}))^{\vartheta} \end{aligned} \tag{31}$$

pins down marginal costs. Given a solution to the system of dynamic equations augmented by the stochastic processes (**Equations 15**, **16**, and **17**), the general equilibrium policy functions (as a function of relevant state variables) can be obtained from (30).

In fact, we are looking for a yet unknown function $mc(\mathbb{Z}_t; \mathbb{Y}_t)$ which simultaneously solves all equilibrium conditions and the maximized Bellman equation. Observe that the costate variable (31) depends on the stochastic shocks and price dispersion. This is why our approach to solve for the general equilibrium values has been to augment the vector of state variables of the household's value function by the law of motions for expectations $x_{1,t}$ and $x_{2,t}$, price dispersion v_t. In Section 6, we use the static condition (31) recursively to pin down marginal costs in general equilibrium.[3] In particular, this approach solves for the household's HJB equation, given aggregate equilibrium dynamics.

Impulse response. To compute impulse response functions, we initialize a shock to the system and solve the resulting system of ODEs using the Relaxation algorithm (Trimborn, Koch, and Steger, 2008). For stochastic simulations, we may add stochastic processes and make use of the policy functions obtained before (cf. Posch and Trimborn, 2011).[4]

[3] The traditional approach is to (log-)linearize the equilibrium conditions and solve the system of linear dynamic equations. In contrast, our non-linear approach uses the recursive competitive solution and the implied general equilibrium value function, which in turn pins down the unknown marginal costs.

[4] We may use the Relaxation algorithm to solve the dynamic equilibrium (non-linear) system. As long as the derivatives of the unknown policy function do *not* appear in the deterministic system, we may obtain the solution without any recursion. Otherwise, we may use the Waveform Relaxation algorithm.

Table 1: Parameterization

ϑ	1	Frisch labor supply elasticity
ρ	0.01	subjective rate of time preference, $\rho = -4 \log 0.9975$
ψ	1	preference for leisure
δ	0.65	Calvo parameter for probability of firms receiving signal, $\delta = -4 \log 0.85$
ε	25	elasticity of substitution intermediate goods
s_g	0.05	share of government consumption
ρ_d	0.4214	autoregressive component preference shock, $\rho_d = -4 \log 0.9$
ρ_a	0.4214	autoregressive component technology shock, $\rho_a = -4 \log 0.9$
ρ_g	0.4214	autoregressive component government shock, $\rho_g = -4 \log 0.9$
σ_d	0.1053	variance preference shock, $\sigma_d^2/(2\rho_d) = 0.025^2/(1-0.9^2)$
σ_a	0.1053	variance technology shock, $\sigma_a^2/(2\rho_a) = 0.025^2/(1-0.9^2)$
σ_g	0.1053	variance government shock, $\sigma_g^2/(2\rho_g) = 0.025^2/(1-0.9^2)$
σ_m	0.025	variance monetary policy shock
θ_1	2	inflation response Taylor rule
θ_2	0.5	interest rate response Taylor rule
π_{ss}	0.005	steady-state inflation

5.1. The equilibrium dynamics at the zero lower bound

This section reports the impulse responses based on the non-linear equilibrium dynamics for the parameterization summarized in Table 1. In particular, we are interested in the effect of the zero lower bound (ZLB) for various shocks. For this objective, we initialize $r_0 = 0.5\%$ slightly higher than the U.S. Federal Funds Interest rate, currently at 0.2%, and then analyze shocks that drive the interest rate towards the ZLB.[5]

5.1.1. A monetary policy shock

Figures 1 and 2 illustrate the effect of a monetary policy shock. On impact, the negative impulse to the interest rate drives the (shadow) interest rate below the ZLB. As the constraint is binding, on impact the positive response of consumption and hours is not as pronounced, the equilibrium marginal cost does not increase as much, while the inflation response is higher. The adjustment towards the equilibrium marginal costs generally is more sluggish. After roughly four quarters, the ZLB is no longer binding and variables move towards their equilibrium values.

5.1.2. A preference shock

Figures 3 and 4 illustrate the effect of a preference shock. Though the ZLB by construction does not bind on impact, the fact that exactly this is anticipated to happen leads to a larger effect on hours and consumption on impact. In what follows, the negative impulse to preferences drives the (shadow) interest rate below the ZLB after roughly one quarter. As

[5]Our parameterization corresponds to the discrete-time model (cf. Section A.3) and roughly coincides with plausible values for the U.S. economy with steady state interest rate of 1.5%.

now the constraint is binding, the response of consumption and hours slightly overshoots after roughly ten quarters. The inflation response on impact is higher, but returns faster to its equilibrium value compared to a situation where we allow for negative interest rates. The non-linear effect of the ZLB on the dynamics for price dispersion in this economy is clearly visible in Figure 3. The adjustment towards the equilibrium marginal costs generally is more sluggish. Finally, after roughly 3 years (12 quarters), the ZLB does no longer bind and the variables gradually move towards their equilibrium values.

5.1.3. A productivity shock

Figures 5 and 6 illustrate the effect of a productivity shock. Except for the response for consumption, the effects of the ZLB on the economic aggregates of a positive productivity shock are roughly comparable to those of a negative preference shock (see above). As we show in Figure 6, optimal consumption increases on impact but not as much as without the presence of the ZLB (while the negative effect on hours is more pronounced). Moreover, the economy escapes the ZLB slightly earlier after roughly 10 quarters.

5.1.4. A government spending shock

Figures 7 and 8 illustrate the effect of a government spending shock in the form of a more restrictive fiscal policy. We do not find any effects of the ZLB on economic dynamics since the constraint is never binding (and by construction does not bind in on impact). This is remarkable since restrictive fiscal policy typically lowers the interest rate. However, since the dynamics of the interest dynamics to return to its steady-state are much stronger such that households and firms anticipate that the ZLB does not bind at any point. This result is robust to higher values for the share of government consumption.

6. Numerical Solution in the policy function space

In what follows, we solve the concentrated HJB equation (26) using a collocation method based on the Matlab `CompEcon` toolbox (Miranda and Fackler 2002). Define the state space $U_z \subseteq \mathbb{R}^n$ and the control region $U_x \subseteq \mathbb{R}^m$. We may write the control problem as

$$\rho V(\mathbb{Z}_t; \mathbb{Y}_t) = f(\mathbb{Z}_t, \mathbb{X}_t) + g(\mathbb{Z}_t, \mathbb{X}_t)^\top V_{\mathbb{Z}} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t, \mathbb{X}_t)\sigma(\mathbb{Z}_t, \mathbb{X}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) \tag{32}$$

in which $\mathbb{Z}_t \in U_z$ denotes the n-vector of states, $\mathbb{X}_t \in U_x$ denotes the m-vector of controls, and $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$ is determined in general equilibrium as a function of the state variables. We define the *reward function* $f : U_z \times U_x \to \mathbb{R}$, the *drift function* $g : U_z \times U_x \to \mathbb{R}^n$, the *diffusion function* $\sigma : U_z \times U_x \to \mathbb{R}^{n\times k}$, $V_{\mathbb{Z}}$ is an n-vector and $V_{\mathbb{Z}\mathbb{Z}}$ is a $n \times n$ matrix, and

$\text{tr}(\mathbb{A})$ denotes the trace of the square matrix $\mathbb{A}$. In general, the state equation follows

$$\mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t + \sigma(\mathbb{Z}_t, \mathbb{X}_t)\,\mathrm{d}B_t$$

where B_t is an k-vector of k independent standard Brownian motions. The instantaneous covariance of $\mathbb{Z}_t$ is $\sigma(\mathbb{Z}_t, \mathbb{X}_t)\sigma(\mathbb{Z}_t, \mathbb{X}_t)^\top$, which may be less than full rank.

First, the first-order conditions (24) and (25) yield *optimal controls* as a function of the states and costate variables:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \equiv \begin{bmatrix} c(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \\ l(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \end{bmatrix} = \begin{bmatrix} (V_a(\mathbb{Z}_t; \mathbb{Y}_t))^{-1} d_t \\ (V_a(\mathbb{Z}_t; \mathbb{Y}_t) w_t/(d_t\psi))^{1/\vartheta} \end{bmatrix}.$$

Second, we define the *reward function* as a function of the states and the optimal controls:

$$f(\mathbb{Z}_t, \mathbb{X}_t) = d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}.$$

Third, we define the *drift function* of the state transition equations:

$$g(\mathbb{Z}_t, \mathbb{X}_t) = \begin{bmatrix} (r_t - \pi_t) a_t - c_t + w_t l_t + T_t + \digamma_t \\ \theta_0 + \theta_1 \pi_t - \theta_2 r_t \\ \delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t \\ (\delta - (\varepsilon - 1)\pi_t) x_{1,t} - d_t/(1 - s_g s_{g,t}) \\ (\delta - \varepsilon \pi_t)\, x_{2,t} - mc_t d_t/(1 - s_g s_{g,t}) \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2) d_t \\ -(\rho_a \log A_t - \frac{1}{2}\sigma_a^2) A_t \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2) s_{g,t} \end{bmatrix}.$$

Fourth, we define the *diffusion function* of the state transition equations:

$$\sigma(\mathbb{Z}_t, \mathbb{X}_t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_d d_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_a A_t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_g s_{g,t} \end{bmatrix}.$$

As an initial guess we may use the solution of the linear-quadratic problem.[6] Since

[6]Note that for an initial guess we may use the policy function as a time-invariant function of the state variables (and auxiliary variables, which in fact are functions of the state variables) obtained from either the non-linear equilibrium dynamics, a (log)linear approximation, or the linear-quadratic problem.

Table 2: Summary of the solution algorithm in the policy function space

Step 1	(Initialization)	Provide an initial guess for the coefficients for a given set of collocation nodes and basis functions.
Step 2	(Solution)	Compute the optimal value of the controls for the set of nodal values for the state and costate variables.
Step 3	(Update)	Update the value function coefficients.
Step 4	(Iteration)	Repeat Steps 2 and 3 until convergence.

the functional form of the solution is unknown, our basic strategy for solving the HJB equation is to approximate $V(\mathbb{Z}_t;\mathbb{Y}_t) \approx \phi(\mathbb{Z}_t;\mathbb{Y}_t)v$, in which v is an n-vector of coefficients and ϕ is the $n \times n$ basis matrix. Starting from the concentrated HJB equation (32), our initial guess for the coefficients from an approximation of the value function and/or control variables for given set of collocation nodes and basis functions $\phi(\mathbb{Z}_t;\mathbb{Y}_t)$ reads:

$$\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t)v = f(\mathbb{Z}_t,\mathbb{X}_t) + g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t)v + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t,\mathbb{X}_t)\sigma(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}v\right)$$

or

$$v = \left(\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t) - g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t) - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t,\mathbb{X}_t)\sigma(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}\right)\right)^{-1} f(\mathbb{Z}_t,\mathbb{X}_t).$$

Using the coefficients we compute the optimal value of the controls for the set of nodal values for the states, which in turn is used to update the value function coefficients. We iterate computing controls and updating the coefficients until convergence (cf. Table 2).

It illustrates the recursive nature of the problem: households take as given *aggregate* variables $\mathbb{Y}_t$ which, however, depend on households' decisions. We start the recursion using $\mathbb{Y}_t^{(0)} = \mathbb{Y}_{ss}$, then update the vector of aggregate variables $\mathbb{Y}_t^{(i)} = \mathbb{Y}(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t)^{(i-1)})$ for $i = 1, \dots$ until convergence. If necessary, we solve this recursive problem by adding a state variable for government liabilities.[7] Finally, we impose general equilibrium by considering optimal policy functions $c(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t))$, $l(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t))$.

Impulse response. Given our solution $V(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t)) \approx \phi(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t)v$, we may simulate the optimal values of the controls for a set of nodal values for the state variables. We may use the Matlab `CompEcon` toolbox to conduct a Monte Carlo simulation of a (controlled) multidimensional Itô processes. In particular, we obtain impulse response functions by setting $\sigma(\mathbb{Z}_t,\mathbb{X}_t) = 0$ and initializing the impulse for one of the states.

[7] This is particularly important for cases where aggregate variables depend on government liabilities.

7. Results

[to be completed]

8. Estimation

[to be completed]

9. Concluding Remarks

[to be completed]

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Figure 1: Impulse responses to a monetary policy shock

In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to the interest rate and its effect to the price dispersion, marginal cost, inflation, the preference shock, the technology shock and the government expenditure shock based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.

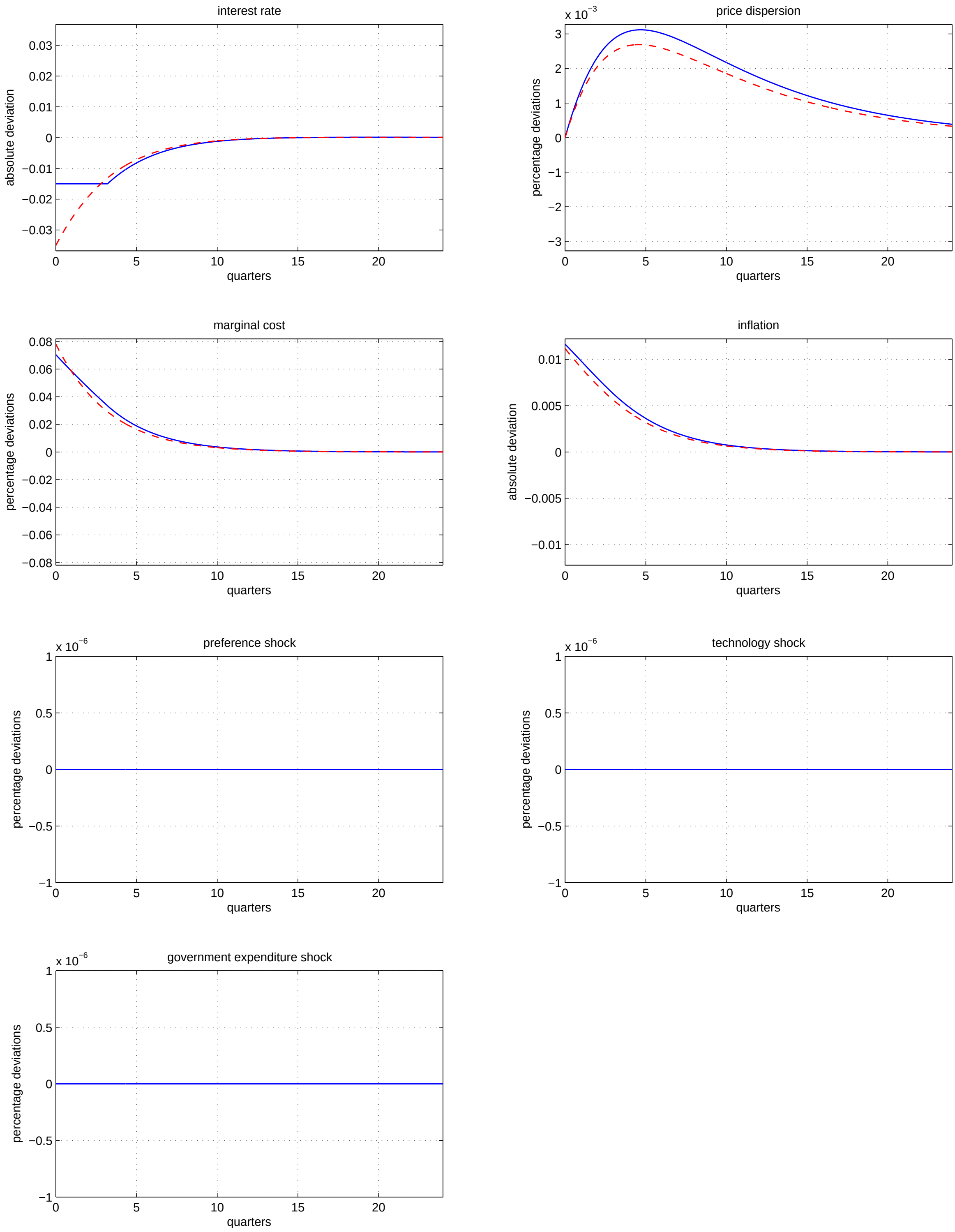


Figure 2: Impulse responses to a monetary policy shock

In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to the interest rate and its effect on optimal consumption, the auxiliary variable for expected revenues, optimal hours, and the auxiliary variable for expected cost based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.

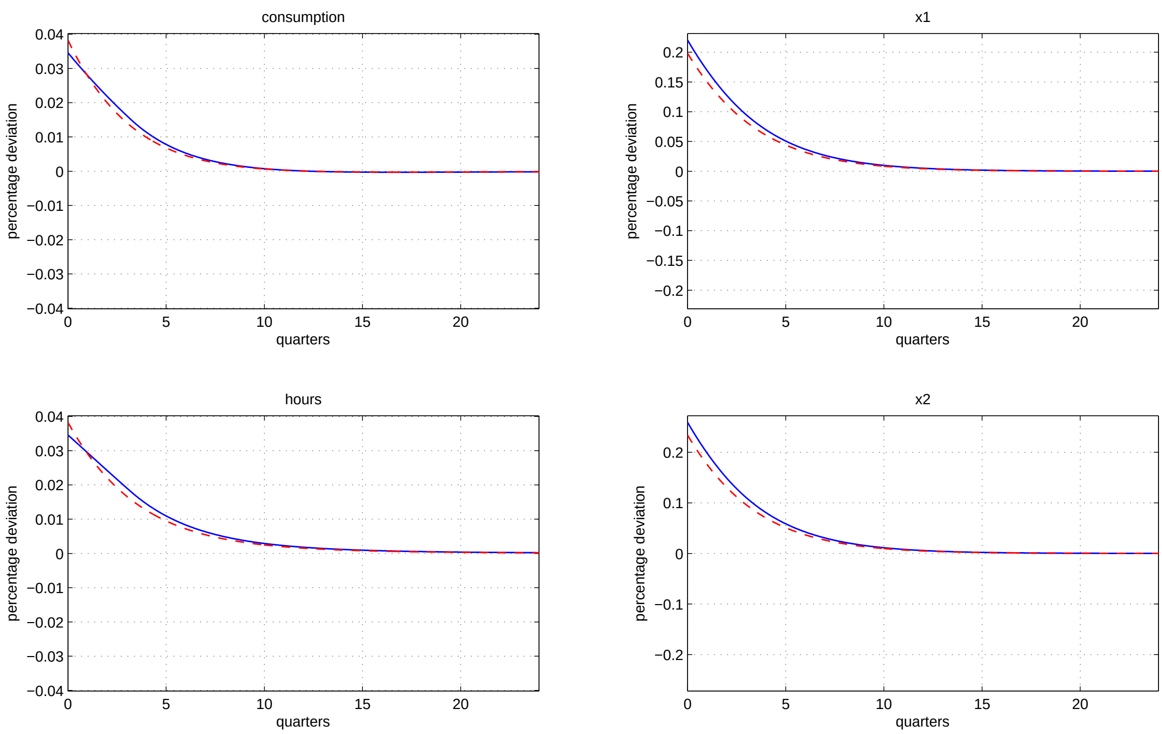


Figure 3: Impulse responses to a preference shock
In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to preferences and its effect to the interest rate, price dispersion, marginal cost, inflation, the technology shock and the government expenditure shock based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.

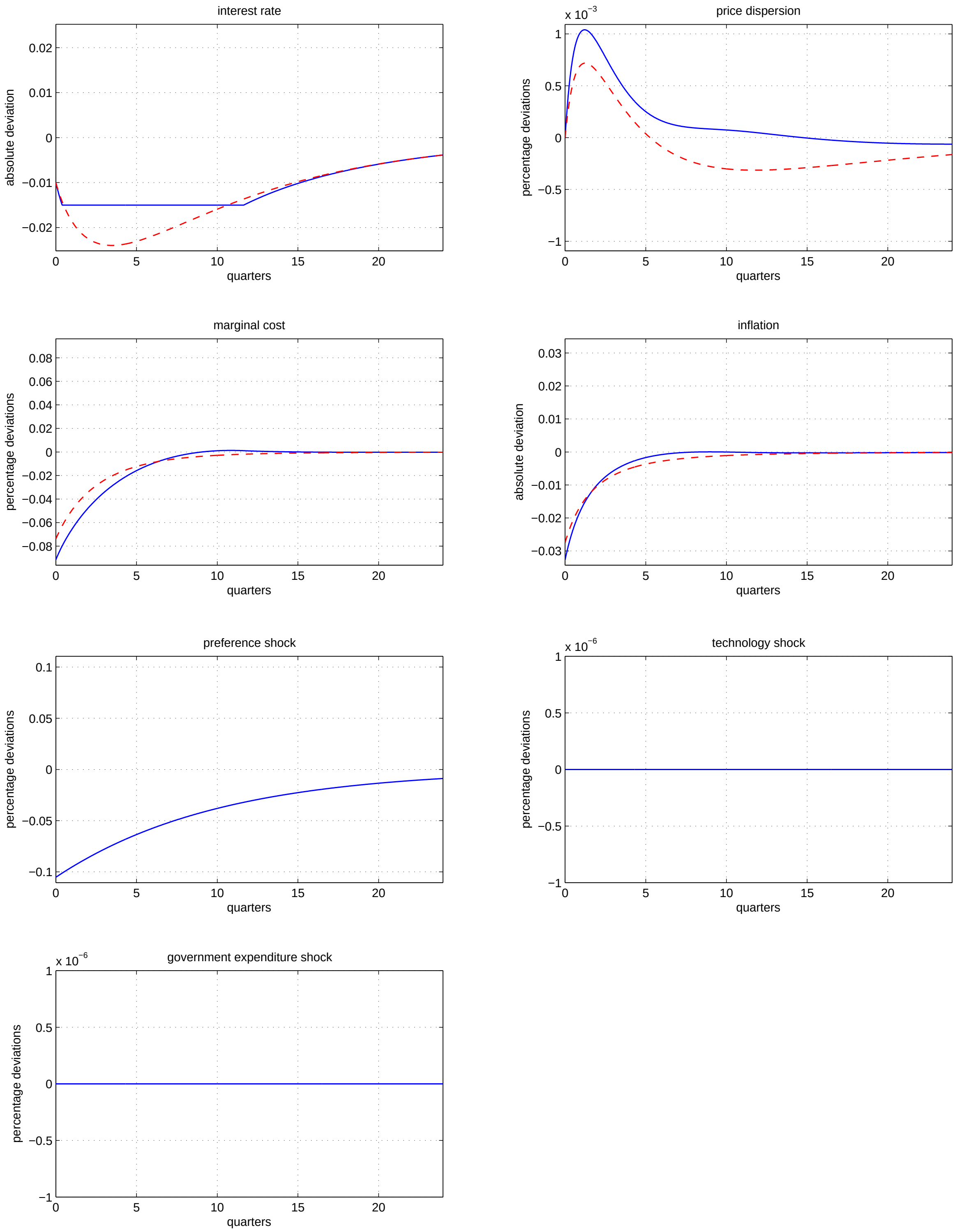


Figure 4: Impulse responses to a preference shock

In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to preferences and its effect on optimal consumption, the auxiliary variable for expected revenues, optimal hours, and the auxiliary variable for expected cost based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.

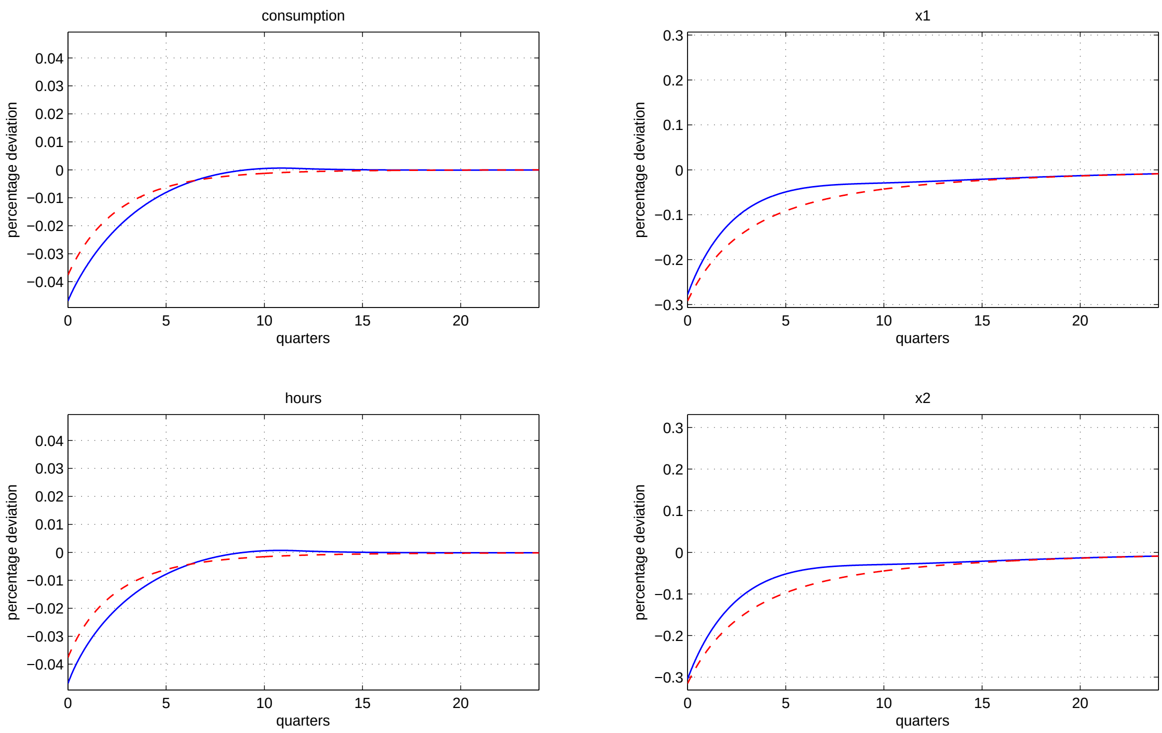


Figure 5: Impulse responses to a productivity shock

In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to technology and its effect to the interest rate, price dispersion, marginal cost, inflation, the preference shock and the government expenditure shock based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.

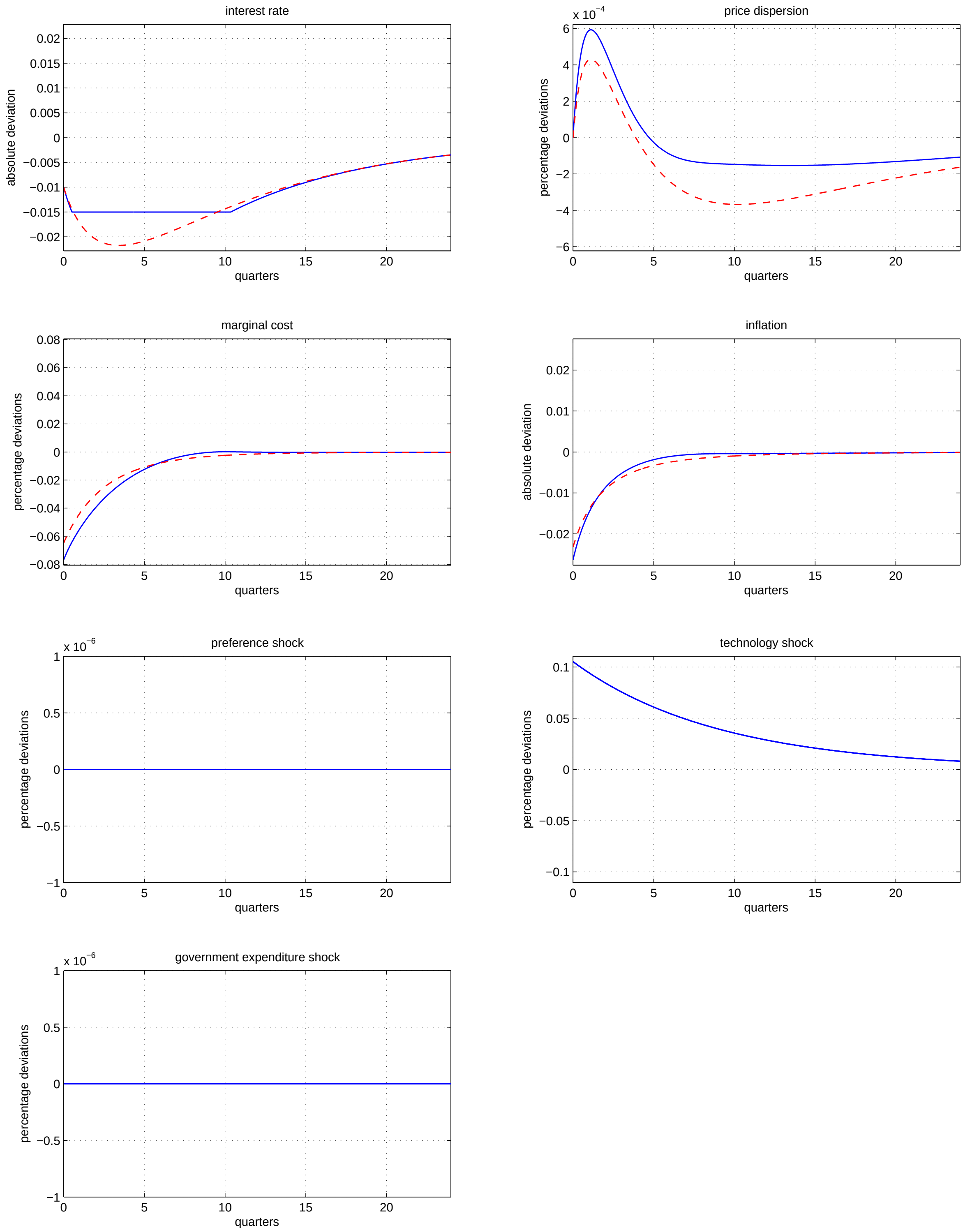


Figure 6: Impulse responses to a productivity shock

In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to technology and its effect on optimal consumption, the auxiliary variable for expected revenues, optimal hours, and the auxiliary variable for expected cost based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.

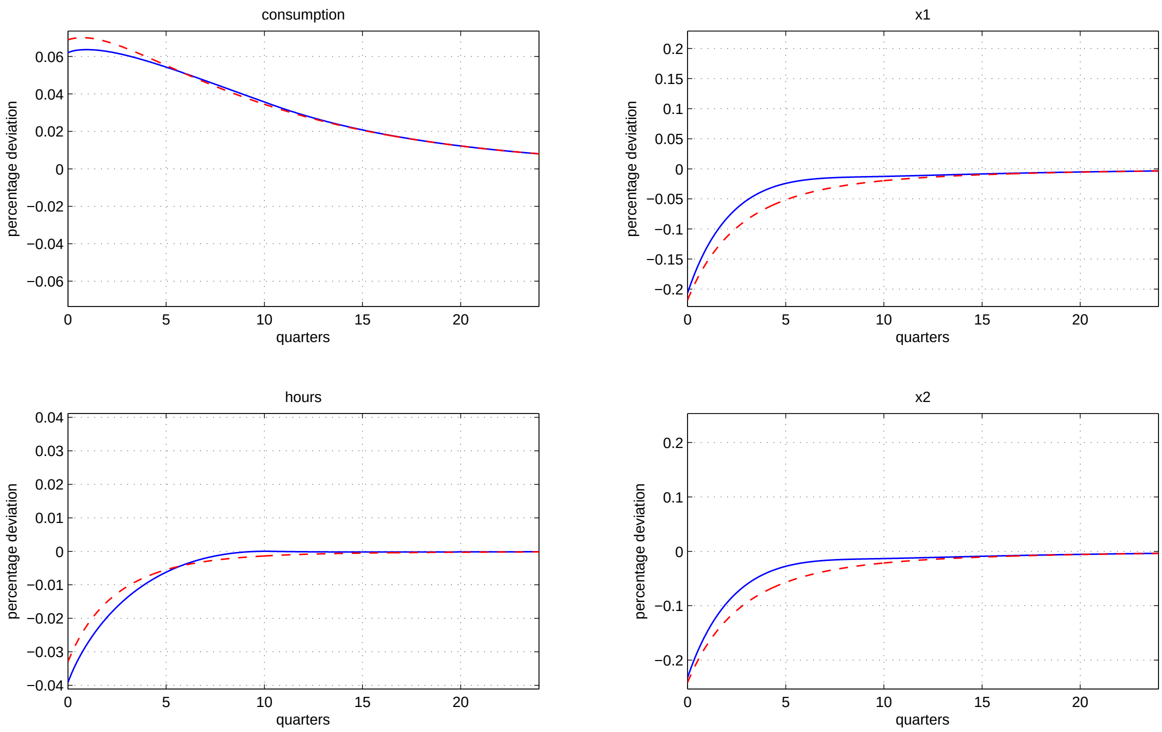


Figure 7: Impulse responses to a government spending shock

In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to government expenditure and its effect to the interest rate, price dispersion, marginal cost, inflation, the preference shock and the technology shock based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.

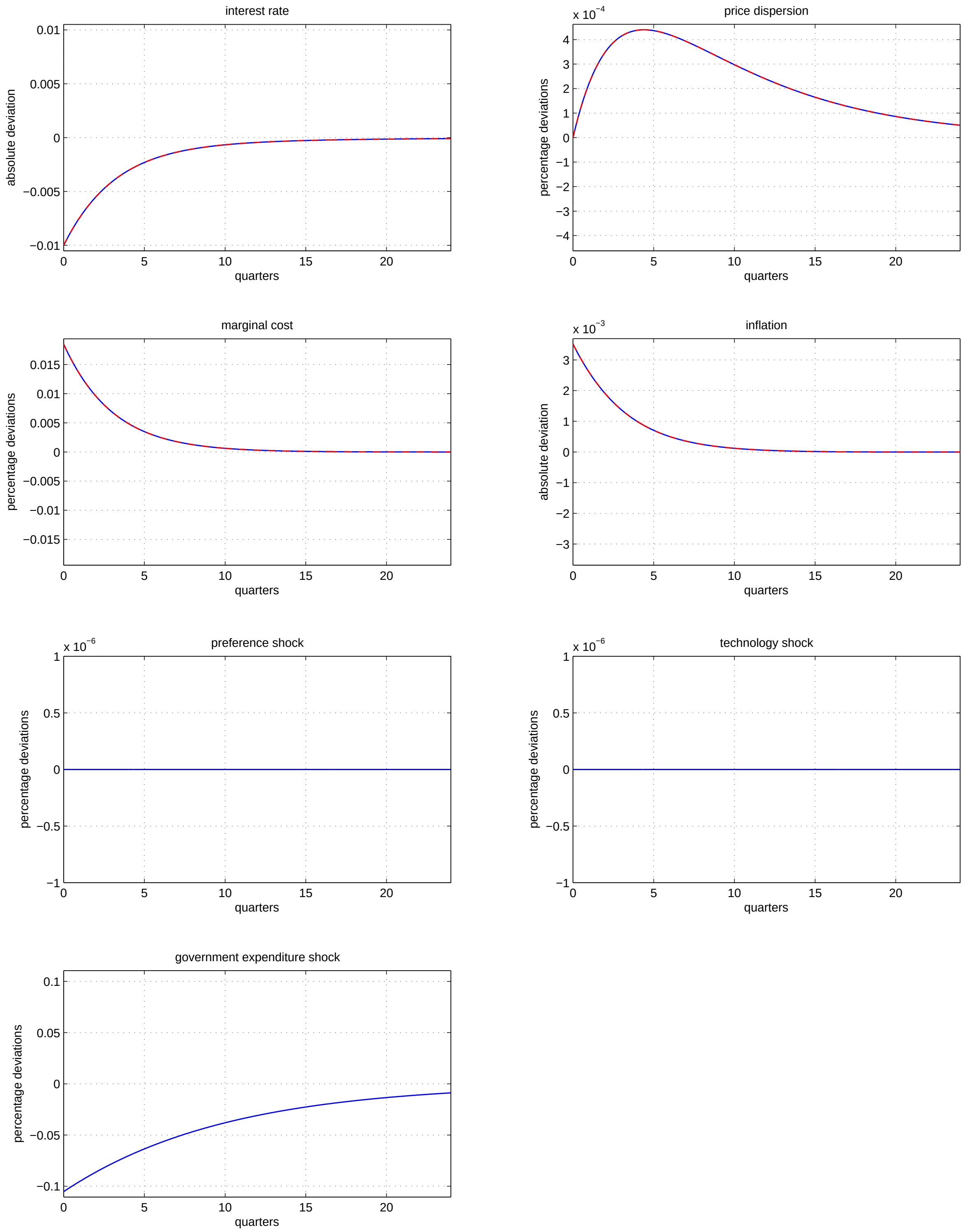
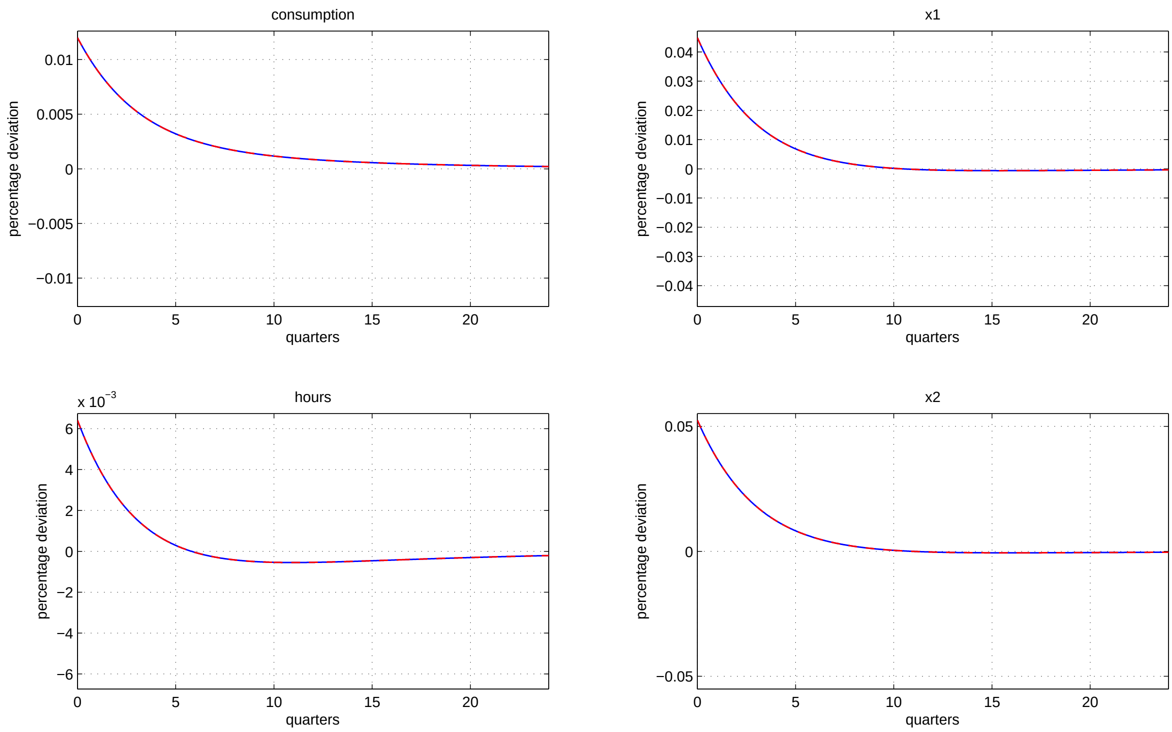


Figure 8: Impulse responses to a government spending shock
In this figure we show (from left to right, top to bottom) the simulated responses for an impulse to government expenditure and its effect on optimal consumption, the auxiliary variable for expected revenues, optimal hours, and the auxiliary variable for expected cost based on the non-linear equilibrium dynamics. While the blue solid line considers the zero lower bound, the red dashed line allows for negative values.



A. Appendix

A.1. Linear-Quadratic Control

Typically, linear and quadratic approximants of g and f are constructed by forming the first- and second-order Taylor expansion around the steady-state $(\mathbb{Z}^*, \mathbb{X}^*)$,

$$\begin{aligned}
f(\mathbb{Z}_t, \mathbb{X}_t) &\approx f^* + f_{\mathbb{Z}}^{*\top}(\mathbb{Z}_t - \mathbb{Z}^*) + f_{\mathbb{X}}^{*\top}(\mathbb{X}_t - \mathbb{X}^*) + \tfrac{1}{2}(\mathbb{Z}_t - \mathbb{Z}^*)^\top f_{\mathbb{Z}\mathbb{Z}}^*(\mathbb{Z}_t - \mathbb{Z}^*) \\
&\quad + (\mathbb{Z}_t - \mathbb{Z}^*)^\top f_{\mathbb{Z}\mathbb{X}}^*(\mathbb{X}_t - \mathbb{X}^*) + \tfrac{1}{2}(\mathbb{X}_t - \mathbb{X}^*)^\top f_{\mathbb{X}\mathbb{X}}^*(\mathbb{X}_t - \mathbb{X}^*), \\
g(\mathbb{Z}_t, \mathbb{X}_t) &\approx g^* + g_{\mathbb{Z}}^*(\mathbb{Z}_t - \mathbb{Z}^*) + g_{\mathbb{X}}^*(\mathbb{X}_t - \mathbb{X}^*),
\end{aligned}$$

where $f^*, g^*, f_{\mathbb{Z}}^*, f_{\mathbb{X}}^*, g_{\mathbb{Z}}^*, g_{\mathbb{X}}^*, f_{\mathbb{Z}\mathbb{Z}}^*, f_{\mathbb{Z}\mathbb{X}}^*$, and $f_{\mathbb{X}\mathbb{X}}^*$ are the values and partial derivatives of f and g evaluated at the steady-state. Thus, the approximate solution to the general problem is supposed to capture the local dynamics around that stationary value.

Collecting terms, we may identify the coefficient of the linear-quadratic problem as

$$\begin{aligned}
\tilde{f}_0 &\equiv f^* - f_{\mathbb{Z}}^{*\top}\mathbb{Z}^* - f_{\mathbb{X}}^{*\top}\mathbb{X}^* + \tfrac{1}{2}\mathbb{Z}^{*\top} f_{\mathbb{Z}\mathbb{Z}}^*\mathbb{Z}^* + \mathbb{Z}^{*\top} f_{\mathbb{Z}\mathbb{X}}^*\mathbb{X}^* + \tfrac{1}{2}\mathbb{X}^{*\top} f_{\mathbb{X}\mathbb{X}}^*\mathbb{X}^*, \\
\tilde{f}_{\mathbb{Z}} &\equiv f_{\mathbb{Z}}^* - f_{\mathbb{Z}\mathbb{Z}}^*\mathbb{Z}^* - f_{\mathbb{Z}\mathbb{X}}^*\mathbb{X}^*, \\
\tilde{f}_{\mathbb{X}} &\equiv f_{\mathbb{X}}^* - f_{\mathbb{Z}\mathbb{X}}^{*\ \top}\mathbb{Z}^* - f_{\mathbb{X}\mathbb{X}}^*\mathbb{X}^*, \\
\tilde{f}_{\mathbb{Z}\mathbb{Z}} &\equiv f_{\mathbb{Z}\mathbb{Z}}^*, \quad \tilde{f}_{\mathbb{Z}\mathbb{X}} \equiv f_{\mathbb{Z}\mathbb{X}}^*, \quad \tilde{f}_{\mathbb{X}\mathbb{X}} \equiv f_{\mathbb{X}\mathbb{X}}^*, \\
\tilde{g}_0 &\equiv g^* - g_{\mathbb{Z}}^*\mathbb{Z}^* - g_{\mathbb{X}}\mathbb{X}^*, \quad \tilde{g}_{\mathbb{Z}} \equiv g_{\mathbb{Z}}^*, \quad \tilde{g}_{\mathbb{X}} \equiv g_{\mathbb{X}}^*.
\end{aligned}$$

where

$$\tilde{f}(\mathbb{Z}_t, \mathbb{X}_t) = \tilde{f}_0 + \tilde{f}_{\mathbb{Z}}^\top \mathbb{Z}_t + \tilde{f}_{\mathbb{X}}^\top \mathbb{X}_t + \tfrac{1}{2}\mathbb{Z}_t^\top \tilde{f}_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t + \mathbb{Z}_t^\top \tilde{f}_{\mathbb{Z}\mathbb{X}}\mathbb{X}_t + \tfrac{1}{2}\mathbb{X}_t^\top \tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t, \tag{A.1}$$

and a linear state transition function

$$\tilde{g}(\mathbb{Z}_t, \mathbb{X}_t) = \tilde{g}_0 + \tilde{g}_{\mathbb{Z}}\mathbb{Z}_t + \tilde{g}_{\mathbb{X}}\mathbb{X}_t, \tag{A.2}$$

$\mathbb{Z}_t$ is the $n \times 1$ state vector, $\mathbb{X}_t$ is the $m \times 1$ control vector, and the parameters are $\tilde{f}_0$, a constant; $\tilde{f}_{\mathbb{Z}}$, a $n \times 1$ vector; $\tilde{f}_{\mathbb{X}}$, a $m \times 1$ vector; $\tilde{f}_{\mathbb{Z}\mathbb{Z}}$, an $n \times n$ matrix, $\tilde{f}_{\mathbb{Z}\mathbb{X}}$, an $n \times m$ matrix; $\tilde{f}_{\mathbb{X}\mathbb{X}}$, an $m \times m$ matrix; $\tilde{g}_0$, an $n \times 1$ vector; $\tilde{g}_{\mathbb{Z}}$, an $n \times n$ matrix; and $\tilde{g}_{\mathbb{X}}$, an $n \times m$ matrix.

As from (23), the *HJB equation* for the (non-stochastic) problem reads

$$\rho V(\mathbb{Z}_t) = \max_{\mathbb{X}_t \in U} \left\{ \tilde{f}(\mathbb{Z}_t, \mathbb{X}_t) + V_{\mathbb{Z}}(\mathbb{Z}_t)^\top \tilde{g}(\mathbb{Z}_t, \mathbb{X}_t) \right\}, \tag{A.3}$$

Observe that we have m *first-order conditions*,

$$\tilde{f}_{\mathbb{X}} + \tilde{f}_{\mathbb{Z}\mathbb{X}}^\top \mathbb{Z}_t + \tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t + \tilde{g}_{\mathbb{X}}^\top V_{\mathbb{Z}}(\mathbb{Z}_t) = 0$$

which imply

$$\mathbb{X}_t = -\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top}\mathbb{Z}_t + \tilde{g}_{\mathbb{X}}^{\top}V_{\mathbb{Z}}(\mathbb{Z}_t)\right], \tag{A.4}$$

that is, the controls are linear in the state and the costate variables. We proceed as follows. The solution of the linear-quadratic problem is the value function which satisfies both the maximized HJB equation and the first-order condition. We may guess a value function and derive conditions under which the guess indeed is the solution to our problem.

An educated guess for the value function is

$$V(\mathbb{Z}_t) = \mathbb{C}_0 + \Lambda_{\mathbb{Z}}^{\top}\mathbb{Z}_t + \tfrac{1}{2}\mathbb{Z}_t^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t \tag{A.5}$$

in which the parameters $\mathbb{C}_0$, a constant; $\Lambda_{\mathbb{Z}}$ a $n \times 1$ vector; and $\Lambda_{\mathbb{Z}\mathbb{Z}}$, a $n \times n$ matrix need to be determined. This implies that the costate variable, a $n \times 1$ vector (and thus the controls) is linear in the states

$$V_{\mathbb{Z}}(\mathbb{Z}_t) = \Lambda_{\mathbb{Z}} + \Lambda_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t \tag{A.6}$$

Inserting everything into the *maximized HJB equation* gives

$$\begin{aligned}
\rho\mathbb{C}_0 + \rho\Lambda_{\mathbb{Z}}^{\top}\mathbb{Z}_t + \rho\tfrac{1}{2}\mathbb{Z}_t^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t \quad &= \quad \tilde{f}_0 + \tilde{f}_{\mathbb{Z}}^{\top}\mathbb{Z}_t + \tilde{f}_{\mathbb{X}}^{\top}\mathbb{X}_t + \tfrac{1}{2}\mathbb{Z}_t^{\top}\tilde{f}_{\mathbb{Z}\mathbb{Z}}\mathbb{Z}_t + \mathbb{Z}_t^{\top}\tilde{f}_{\mathbb{Z}\mathbb{X}}\mathbb{X}_t + \tfrac{1}{2}\mathbb{X}_t^{\top}\tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t \\
&\quad +\Lambda_{\mathbb{Z}}^{\top}\left[\tilde{g}_0 + \tilde{g}_{\mathbb{Z}}\mathbb{Z}_t + \tilde{g}_{\mathbb{X}}\mathbb{X}_t\right] + \mathbb{Z}_t^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\left[\tilde{g}_0 + \tilde{g}_{\mathbb{Z}}\mathbb{Z}_t + \tilde{g}_{\mathbb{X}}\mathbb{X}_t\right]
\end{aligned}$$

where the vector components are:

$$\begin{aligned}
\mathbb{X}_t \quad &= \quad -\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}} + \left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right]\mathbb{Z}_t\right], \\
\mathbb{X}_t^{\top}\tilde{f}_{\mathbb{X}\mathbb{X}}\mathbb{X}_t \quad &= \quad \left[\tilde{f}_{\mathbb{X}}^{\top} + \Lambda_{\mathbb{Z}}^{\top}\tilde{g}_{\mathbb{X}}\right]\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}}\right] + 2\left[\tilde{f}_{\mathbb{X}}^{\top} + \Lambda_{\mathbb{Z}}^{\top}\tilde{g}_{\mathbb{X}}\right]\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right]\mathbb{Z}_t \\
&\quad +\mathbb{Z}_t^{\top}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}_t}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right]^{\top}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right]\mathbb{Z}_t
\end{aligned}$$

Finally equating terms with equal powers determines our coefficients recursively

$$\begin{aligned}
\rho\mathbb{C}_0 \quad &= \quad \tilde{f}_0 - \tilde{f}_{\mathbb{X}}^{\top}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}}\right] + \tfrac{1}{2}\left[\tilde{f}_{\mathbb{X}}^{\top} + \Lambda_{\mathbb{Z}}^{\top}\tilde{g}_{\mathbb{X}}\right]\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}}\right] \\
&\quad +\Lambda_{\mathbb{Z}}^{\top}\left[\tilde{g}_0 - \tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}}\right]\right], \\
\rho\Lambda_{\mathbb{Z}}^{\top} \quad &= \quad \tilde{f}_{\mathbb{Z}}^{\top} - \left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}}\right]^{\top}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \Lambda_{\mathbb{Z}}^{\top}\tilde{g}_{\mathbb{Z}} + \tilde{g}_0^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}} - \left[\tilde{f}_{\mathbb{X}} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}}\right]^{\top}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}, \\
\rho\tfrac{1}{2}\Lambda_{\mathbb{Z}\mathbb{Z}} \quad &= \quad \tfrac{1}{2}\tilde{f}_{\mathbb{Z}\mathbb{Z}} - \tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right] + \tfrac{1}{2}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right]^{\top}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right] \\
&\quad +\tfrac{1}{2}\Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{Z}} + \tfrac{1}{2}\tilde{g}_{\mathbb{Z}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}} - \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^{\top} + \tilde{g}_{\mathbb{X}}^{\top}\Lambda_{\mathbb{Z}\mathbb{Z}}\right].
\end{aligned}$$

Rewriting the last condition gives

$$
\begin{aligned}
0 \;=\;& \tilde{f}_{\mathbb{Z}\mathbb{Z}} + \Lambda_{\mathbb{Z}\mathbb{Z}}(\tilde{g}_{\mathbb{Z}} - \tfrac{1}{2}\rho\mathbb{I}_n) + (\tilde{g}_{\mathbb{Z}} - \tfrac{1}{2}\rho\mathbb{I}_n)^\top \Lambda_{\mathbb{Z}\mathbb{Z}} \\
& - \left[\tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top\Lambda_{\mathbb{Z}\mathbb{Z}} + \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top\Lambda_{\mathbb{Z}\mathbb{Z}}\right],
\end{aligned}
$$

where $\mathbb{I}_n$ is the identity matrix, which gives $\Lambda_{\mathbb{Z}\mathbb{Z}}$ and thus recursively, $\Lambda_{\mathbb{Z}}$ and $\mathbb{C}_0$ as a solution of an algebraic *Ricatti* equation. The vector of coefficients $\Lambda_{\mathbb{Z}}$ is obtained from

$$
\Lambda_{\mathbb{Z}} = \left[\rho\mathbb{I}_n + \tilde{f}_{\mathbb{Z}\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top + \Lambda_{\mathbb{Z}\mathbb{Z}}\tilde{g}_{\mathbb{X}}\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\tilde{g}_{\mathbb{X}}^\top - \tilde{g}_{\mathbb{Z}}^\top\right]^{-1}\left[\tilde{f}_{\mathbb{Z}}^\top - \tilde{f}_{\mathbb{X}}^\top\tilde{f}_{\mathbb{X}\mathbb{X}}^{-1}\left[\tilde{f}_{\mathbb{Z}\mathbb{X}}^\top + \tilde{g}_{\mathbb{X}}^\top\Lambda_{\mathbb{Z}\mathbb{Z}}\right] + \tilde{g}_0^\top\Lambda_{\mathbb{Z}\mathbb{Z}}\right]^\top.
$$

This closes the proof that the guess indeed is the solution.

A.2. Linear approximations

In order to analyze local dynamics, the traditional approach is to approximate the dynamic equilibrium system around steady-state values. We define we $\hat{x}_t \equiv (x_t - x_{ss})/x_{ss}$, where x_{ss} is the steady-state value for the variable x_t. Thus, we can write $x_t = (1 + \hat{x}_t)x_{ss}$.

- Euler equation, the first-order conditions of the household, and budget constraint:

Equation 1

$$
\mathbb{E}_t(\mathrm{d}\hat{\lambda}_t) = (-(\delta - (\varepsilon - 1)\pi_{ss})\lambda_{ss}\hat{x}_{1,t} + (\delta - (\varepsilon - 1)\pi_{ss})\lambda_{ss}\hat{x}_{2,t} - r_{ss}\lambda_{ss}\hat{r}_t)\,\mathrm{d}t
$$

Equation 2

$$
\vartheta\hat{l}_t + \hat{c}_t = \hat{w}_t
$$

Equation 3

$$
\hat{d}_t - \hat{c}_t = \hat{\lambda}_t
$$

(redundant)

$$
\mathrm{d}\hat{a}_t = 0
$$

- Profit maximization is given by:

Equation 4

$$\hat{\Pi}^*_t = \hat{x}_{2,t} - \hat{x}_{1,t}$$

Equation 5

$$\mathrm{d}\hat{x}_{1,t} = \Big(\varepsilon(\delta - (\varepsilon-1)\pi_{ss})x_{1,ss}\hat{x}_{1,t} - (\varepsilon-1)(\delta - (\varepsilon-1)\pi_{ss})x_{1,ss}\hat{x}_{2,t} \\ -\lambda_{ss}y_{ss}(\hat{\lambda}_t + \hat{y}_t)\Big)\mathrm{d}t$$

Equation 6

$$\mathrm{d}\hat{x}_{2,t} = \Big(\varepsilon(\delta - (\varepsilon-1)\pi_{ss})x_{2,ss}\hat{x}_{1,t} + ((1-\varepsilon)(\delta - \varepsilon\pi_{ss}) - \varepsilon\pi_{ss})x_{2,ss}\hat{x}_{2,t} \\ -\lambda_{ss}y_{ss}mc_{ss}(\hat{\lambda}_t + \hat{y}_t + \hat{mc}_t)\Big)\mathrm{d}t$$

Equation 7

$$(1 - mc_{ss}v_{ss})\hat{F}_t = -mc_{ss}v_{ss}(\hat{mc}_t + \hat{v}_t) + (1 - mc_{ss}v_{ss})\hat{y}_t$$

Equation 8

$$\hat{w}_t = \hat{A}_t + \hat{mc}_t$$

- Government policy:

Equation 9

$$\mathrm{d}\hat{r}_t = (-\theta_1(\delta - (\varepsilon-1)\pi_{ss})\hat{x}_{1,t} + \theta_1(\delta - (\varepsilon-1)\pi_{ss})\hat{x}_{2,t} - \theta_2 r_{ss}\hat{r}_t)\,\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}$$

Equation 10

$$\hat{g}_t = \hat{s}_{g,t} + \hat{y}_t$$

(redundant)

$$T_{ss}\hat{T}_t = -r_{ss}a_{ss}(\hat{r}_t + \hat{a}_t) - s_g y_{ss}(\hat{s}_{g,t} + \hat{y}_t)$$

- Inflation evolution and price dispersion:

Equation 11

$$\Pi_{ss}\hat{\Pi}_t = \delta\left(\Pi^*_{ss}\right)^{1-\varepsilon}\hat{\Pi}^*_t, \quad \text{for} \quad \Pi_t \equiv 1 + \pi_t$$

Equation 12

$$\mathrm{d}\hat{v}_t = \Big(\varepsilon\pi_{ss}v_{ss}\hat{\Pi}^*_t + (\varepsilon\pi_{ss} - \delta)v_{ss}\hat{v}_t\Big)\mathrm{d}t$$

- Market clearing on goods and labor markets:

  Equation 13

$$y_{ss}(\hat{y}_t - \hat{g}_t) = c_{ss}(\hat{c}_t - \hat{g}_t) \quad \text{(expenditure)}$$

  Equation 14

$$\hat{y}_t = \hat{A}_t + \hat{l}_t - \hat{v}_t \quad \text{(production)}$$

  (redundant)

$$y_{ss}\hat{y}_t = w_{ss}l_{ss}(\hat{w}_t + \hat{l}_t) + \digamma_{ss}\hat{\digamma}_t \quad \text{(income)}$$

- Stochastic processes follow:

  Equation 15

$$\mathrm{d}\hat{d}_t = -\left(\rho_d - \tfrac{1}{2}\sigma_d^2\right)\hat{d}_t\mathrm{d}t + \sigma_d\hat{d}_t\mathrm{d}B_{d,t}$$

  Equation 16

$$\mathrm{d}\hat{A}_t = -(\rho_a - \tfrac{1}{2}\sigma_a^2)\hat{A}_t\mathrm{d}t + \sigma_a\hat{A}_t\mathrm{d}B_{a,t}$$

  Equation 17

$$\mathrm{d}\hat{s}_{g,t} = -\left(\rho_g - \tfrac{1}{2}\sigma_g^2\right)\hat{s}_{g,t}\mathrm{d}t + \sigma_g\hat{s}_{g,t}\mathrm{d}B_{g,t}$$

Recall that from (31) we obtain the linearized static condition

$$\hat{mc}_t = -(1+\vartheta)(\hat{\lambda}_t + \hat{A}_t - \hat{d}_t) + \vartheta\hat{v}_t + \vartheta s_g/(1-s_g)\hat{s}_{g,t}$$

Hence, we may summarize the local equilibrium dynamics around steady-state values as:

$$
\begin{aligned}
\mathrm{d}\hat{x}_{1,t} &= \Big([\varepsilon(\delta-(\varepsilon-1)\pi_{ss})]x_{1,ss}\hat{x}_{1,t} \\
&\quad + [-(\varepsilon-1)(\delta-(\varepsilon-1)\pi_{ss})x_{1,ss}]\,\hat{x}_{2,t} \\
&\quad -[1/(1-s_g)]\hat{d}_t - [s_g/(1-s_g)^2]\hat{s}_{g,t}\Big)\mathrm{d}t \\
\mathrm{d}\hat{x}_{2,t} &= \Big(\,[\varepsilon(\delta-(\varepsilon-1)\pi_{ss})x_{2,ss}]\,\hat{x}_{1,t} \\
&\quad + [(1-\varepsilon)(\delta-\varepsilon\pi_{ss})-\varepsilon\pi_{ss}]\,x_{2,ss}\hat{x}_{2,t} \\
&\quad -[\vartheta/(1-s_g)]\hat{v}_t + [(1+\vartheta)/(1-s_g)](\hat{\lambda}_t+\hat{A}_t) \\
&\quad +[(1+\vartheta-mc_{ss})/(1-s_g)]\hat{d}_t - [(\vartheta+mc_{ss})s_g/(1-s_g)^2]\hat{s}_{g,t}\Big)\mathrm{d}t \\
\mathrm{d}\hat{v}_t &= \Big([-\varepsilon\pi_{ss}v_{ss}]\hat{x}_{1,t} + [\varepsilon\pi_{ss}v_{ss}]\hat{x}_{2,t} + [\varepsilon\pi_{ss}-\delta]v_{ss}\hat{v}_t\Big)\mathrm{d}t \\
\mathrm{d}\hat{r}_t &= \Big([-\theta_1(\delta-(\varepsilon-1)\pi_{ss})]\hat{x}_{1,t} + [\theta_1(\delta-(\varepsilon-1)\pi_{ss})]\hat{x}_{2,t} \\
&\quad -[\theta_2 r_{ss}]\hat{r}_t\Big)\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t} \\
\mathrm{d}\hat{\lambda}_t &= \Big([-(\delta-(\varepsilon-1)\pi_{ss})\lambda_{ss}]\hat{x}_{1,t} + [(\delta-(\varepsilon-1)\pi_{ss})\lambda_{ss}]\hat{x}_{2,t} + [-\lambda_{ss}]r_{ss}\hat{r}_t\Big)\mathrm{d}t \\
&\quad +\sigma_d\left(\left.\frac{\partial\lambda_d}{\partial x_{1,t}}\right|_{ss} x_{1,ss}\hat{x}_{1,t} + \left.\frac{\partial\lambda_d}{\partial x_{2,t}}\right|_{ss} x_{2,ss}\hat{x}_{2,t} + \ldots\right)\mathrm{d}B_{d,t} \\
&\quad +\sigma_a\left(\left.\frac{\partial\lambda_A}{\partial x_{1,t}}\right|_{ss} x_{1,ss}\hat{x}_{1,t} + \left.\frac{\partial\lambda_A}{\partial x_{2,t}}\right|_{ss} x_{2,ss}\hat{x}_{2,t} + \ldots\right)\mathrm{d}B_{a,t} \\
&\quad +\sigma_g\left(\left.\frac{\partial\lambda_s}{\partial x_{1,t}}\right|_{ss} x_{1,ss}\hat{x}_{1,t} + \left.\frac{\partial\lambda_s}{\partial x_{2,t}}\right|_{ss} x_{2,ss}\hat{x}_{2,t} + \ldots\right)\mathrm{d}B_{g,t} \\
&\quad +\sigma_m\left(\left.\frac{\partial\lambda_r}{\partial x_{1,t}}\right|_{ss} x_{1,ss}\hat{x}_{1,t}r_{ss} + \left.\frac{\partial\lambda_r}{\partial x_{2,t}}\right|_{ss} x_{2,ss}\hat{x}_{2,t}r_{ss} + \ldots\right)\mathrm{d}B_{m,t} \\
\mathrm{d}\hat{d}_t &= -(\rho_d - \tfrac{1}{2}\sigma_d^2)\hat{d}_t\mathrm{d}t + \sigma_d\hat{d}_t\mathrm{d}B_{d,t} \\
\mathrm{d}\hat{A}_t &= -(\rho_a - \tfrac{1}{2}\sigma_a^2)\hat{A}_t\mathrm{d}t + \sigma_a\hat{A}_t\mathrm{d}B_{a,t} \\
\mathrm{d}\hat{s}_{g,t} &= -\left(\rho_g - \tfrac{1}{2}\sigma_g^2\right)\hat{s}_{g,t}\mathrm{d}t + \sigma_g\hat{s}_{g,t}\mathrm{d}B_{g,t}
\end{aligned}
$$

in which we define percentage deviations $\hat{x}_t \equiv (x_t - x_{ss})/x_{ss}$.[8]

In order to analyze local dynamics around the non-stochastic steady state, we need to study the eigenvalues of the Jacobian matrix evaluated at the steady state,

[8]Note that we used partial derivatives

$$\frac{\partial\pi_t}{\partial x_{1,t}} = \frac{\delta}{1-\varepsilon}\frac{\partial\left(\Pi_t^*\right)^{1-\varepsilon}}{\partial x_{1,t}} = -\delta\left(\Pi_t^*\right)^{1-\varepsilon}/x_{1,t} = -(\delta-(\varepsilon-1)\pi_{ss})/x_{1,t}$$

$$\frac{\partial\pi_t}{\partial x_{2,t}} = \frac{\delta}{1-\varepsilon}\frac{\partial\left(\Pi_t^*\right)^{1-\varepsilon}}{\partial x_{2,t}} = (\delta-(\varepsilon-1)\pi_{ss})/x_{2,t}$$

$$\frac{\partial\Pi_t^*}{\partial x_{1,t}} = -\Pi_t^*/x_{1,t}, \quad \frac{\partial\Pi_t^*}{\partial x_{2,t}} = \Pi_t^*/x_{2,t}$$

$$\mathrm{d}\begin{bmatrix} \hat{x}_{1,t} \\ \hat{x}_{2,t} \\ \hat{v}_t \\ \hat{r}_t \\ \hat{\lambda}_t \\ \hat{d}_t \\ \hat{A}_t \\ \hat{s}_{g,t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & a_{16} & 0 & a_{18} \\ a_{21} & a_{22} & a_{23} & 0 & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & a_{54} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88} \end{bmatrix} \begin{bmatrix} \hat{x}_{1,t} \\ \hat{x}_{2,t} \\ \hat{v}_t \\ \hat{r}_t \\ \hat{\lambda}_t \\ \hat{d}_t \\ \hat{A}_t \\ \hat{s}_{g,t} \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{41} & 0 & 0 & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} \\ 0 & b_{62} & 0 & 0 \\ 0 & 0 & b_{73} & 0 \\ 0 & 0 & 0 & b_{84} \end{bmatrix} \mathrm{d}B_t$$

where
$$\begin{aligned}
a_{11} &\equiv \varepsilon(\delta - (\varepsilon - 1)\pi_{ss})x_{1,ss} \\
a_{12} &\equiv -(\varepsilon - 1)(\delta - (\varepsilon - 1)\pi_{ss})x_{1,ss} \\
a_{16} &\equiv -1/(1 - s_g) \\
a_{18} &\equiv -s_g/(1 - s_g)^2 \\
a_{21} &\equiv \varepsilon(\delta - (\varepsilon - 1)\pi_{ss})x_{2,ss} \\
a_{22} &\equiv ((1 - \varepsilon)(\delta - \varepsilon\pi_{ss}) - \varepsilon\pi_{ss})x_{2,ss} \\
a_{23} &\equiv -\vartheta/(1 - s_g) \\
a_{25} &\equiv (1 + \vartheta)/(1 - s_g) \\
a_{26} &\equiv (1 + \vartheta - mc_{ss})/(1 - s_g) \\
a_{27} &\equiv (1 + \vartheta)/(1 - s_g) \\
a_{28} &\equiv -(\vartheta + mc_{ss})s_g/(1 - s_g)^2 \\
a_{31} &\equiv -\varepsilon\pi_{ss}v_{ss} \\
a_{32} &\equiv \varepsilon\pi_{ss}v_{ss} \\
a_{33} &\equiv (\varepsilon\pi_{ss} - \delta)v_{ss} \\
a_{41} &\equiv -\theta_1(\delta - (\varepsilon - 1)\pi_{ss}) \\
a_{42} &\equiv \theta_1(\delta - (\varepsilon - 1)\pi_{ss}) \\
a_{44} &\equiv -\theta_2 r_{ss} \\
a_{51} &\equiv -(\delta - (\varepsilon - 1)\pi_{ss})\lambda_{ss} \\
a_{52} &\equiv (\delta - (\varepsilon - 1)\pi_{ss})\lambda_{ss} \\
a_{54} &\equiv -\lambda_{ss} r_{ss} \\
a_{66} &\equiv -(\rho_d - \tfrac{1}{2}\sigma_d^2) \\
a_{77} &\equiv -(\rho_a - \tfrac{1}{2}\sigma_a^2) \\
a_{88} &\equiv -(\rho_g - \tfrac{1}{2}\sigma_g^2)
\end{aligned}$$

and the vector of shocks $\mathrm{d}B_t \equiv [\mathrm{d}B_{m,t}, \mathrm{d}B_{d,t}, \mathrm{d}B_{a,t}, \mathrm{d}B_{g,t}]^\top$

A.3. Calibration of model parameters

Suppose that we want to parameterize the Ornstein-Uhlenbeck process and the first-order autoregressive process:

$$\mathrm{d}x_t = -\rho_x x_t \mathrm{d}t + \sigma_x \mathrm{d}B_t \quad \text{and} \quad \tilde{x}_t = \tilde{\rho}_x \tilde{x}_t + \tilde{\sigma}_x \varepsilon_t, \quad x_0 = \tilde{x}_0. \tag{A.7}$$

B_t is a standard Brownian motion and $\varepsilon_t \sim N(0,1)$. Observe that the solutions are:

$$x_t = x_0 e^{-\rho_x t} + e^{-\rho_x t} \int_0^t e^{\rho_x s}\, \mathrm{d}B_s \quad \text{and} \quad \tilde{x}_t = \tilde{\rho}_x^t x_0 + \tilde{\rho}_x^t \tilde{\sigma}_x \sum_{i=1}^t \tilde{\rho}_x^{-i} \varepsilon_i$$

Let us calibrate ρ_x, given a parametric value for $\tilde{\rho}_x$ at the quarterly frequency, such that the expected value $\mathbb{E}(x_1) = \mathbb{E}(\tilde{x}_4)$, and the variance $Var(x_1) = Var(\tilde{x}_4)$ coincide. It is straightforward to show that $\mathbb{E}(x_1) = e^{-\rho_x} x_0$ and $\mathbb{E}(\tilde{x}_4) = \tilde{\rho}_x^4 x_0$. Hence, we obtain ρ_x:

$$e^{-\rho_x} = \tilde{\rho}_x^4 \quad \Rightarrow \quad \rho_x \equiv -4 \log(\tilde{\rho}_x)$$

Itô isometry:

$$Var(x_1) = \sigma_x^2 e^{-2\rho_x} \int_0^1 e^{2\rho_x s}\, \mathrm{d}t = \frac{\sigma_x^2}{2\rho_x}\left(1 - e^{-2\rho_x}\right)$$

and

$$Var(\tilde{x}_4) = \bar{\sigma}_x^2 \sum_{i=1}^4 \bar{\rho}_x^{(i-1)2}$$

Equating terms implies:

$$\sigma_x^2 \equiv 2 \frac{\rho_x \bar{\sigma}_x^2}{1 - e^{-2\rho_x}} \sum_{i=1}^4 \bar{\rho}_x^{(i-1)2}.$$

As an example, $\tilde{\rho}_x = 0.9$ and $\tilde{\sigma}_x = 0.05$ implies $\rho_x \approx 0.42$ and $\sigma_x \approx 0.11$. Both processes converge to the same limiting distribution, $Var(x) = \sigma_x^2/(2\rho_x) = Var(\tilde{x}) = \tilde{\sigma}_x^2/(1 - \tilde{\rho}_x^2)$.